MAXWELL MIXTURE DISTRIBUTIONS

BY

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To my beloved parents

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LIST OF ABBREVIATIONS

TBE	Time Between Events
MLE	Maximum Likelihood Estimation
PDF	Probability Density Function
CDF	Cumulative Distribution Function
MGF	Moment Generating Function
CF	Characteristic Function
MCQ	Maxwell Cumulative Quantity
MMCQ	Mixture of Maxwell Cumulative Quantity
UCL	Upper Control Limit
LCL	Lower Control Limit
CL	Central Line
UPL	Upper Probability Limit
LPL	Lower Probability Limit
ARL	Average Run Length
CCC	Cumulative Count Control
CQC	Cumulative Quantity Control
CPC	Cumulative Probability Control
VBM	Vertical Boring Machine
EM	Expectation Maximization

ABSTRACT

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Maxwell distribution is studied in this thesis in different degrees. Finite and infinite mixture of *Maxwell* distributions have been investigated along with different properties such as moment generating function, characteristic function, survival function, hazard function, m-th raw moments etc. Some special distributions have also been developed for finite and infinite *Maxwell* mixture distributions such as discrete *uniform* mixture of *Maxwell* distribution, *binomial* mixture of *Maxwell* distribution and *tau square* mixture of *Maxwell* distribution. For parameter estimation, maximum likelihood estimation and method of moments are discussed for these distributions. For application purpose, *Maxwell* mixture distributions have been modeled in the field of statistical process monitoring and reliability engineering.

ملخص الرسالة

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تمت دراسه توزيع ماكسويل فى هذه الاطروحه في درجات مختلفة. تم التحقيق من تويعات خليط ماكسويل المتناهى واللأمتناهي الى جانب الخصائص المختلفة مثل داله توليد العزوم، الداله المميزة، داله زمن البقاء، داله المخاطره، العزم الميمى الخ. كما تم تطوير بعض التوزيعات الخاصة لتوزيعات خليط ماكسويل المتناهى و اللأمتناهى مثل خليط التوزيع المنتظم المتقطع و توزيع ماكسويل، خليط توزيع ذي الحدين و توزيع ماكسويل و خليط توزيع مربع تاو و توزيع ماكسويل. لتقدير المعالم، تم مناقشه طريقة الأمكان الأعظم و طريقه العزوم لهذه التوزيعات. لغرض التطبيق، تم نمذجه توزيعات خليط ماكسويل في مجال مراقبة الإحصائية للجودة و الاعتماديه الهندسية.

CHAPTER ONE

INTRODUCTION

1.1. Background

In the current thesis we investigate *Maxwell* distribution in different flavor. We introduce finite and infinite mixture of *Maxwell* distributions. *k* component mixture of *Maxwell* distribution is presented as an example of finite mixture and *tau square* mixture of *Maxwell* distribution is presented as infinite mixture of *Maxwell* distribution. Different properties for example moment generating function (MGF), characteristic function (CF), moments etc. of the mentioned distributions are discussed. Estimation methods also has been discussed separately. Finally, simulation study and real life example have been discussed.

The *Maxwell* distribution with scale parameter σ has following PDF and CDF

$$f(r;\sigma)=\sqrt{(2/\pi)}\sigma^{-3}r^{2}e^{-\left(r^{2}/2\sigma^{2}\right)};r>0\text{, and} \tag{1.1}$$

$$F(r)=\left(2/\sqrt{\pi}\right)\gamma\left((3/2),\left(r^{2}/2\sigma^{2}\right)\right), \tag{1.2}$$

where $\int_0^u x^{v-1}e^{-\mu x}dx=\mu^{-v}\gamma(v,\mu u)$. (1.3)

Figure 1.1 shows the PDF of *Maxwell* distribution which is skewed in property. For a sample of size *n* consider a set of observations $R_1, R_2, ..., R_n$.Then the likelihood and log-likelihood functions are respectively given by

$$L(\sigma;r)=\left(\sqrt{(2/\pi)}\sigma^{-3}\right)^n\prod_{i=1}^n r_i^2 e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n r_i^2}, \text{ and} \tag{1.4}$$

$$\log(L)=n\log\left(\sqrt{\frac{2}{\pi}}\right)-3n\log(\sigma)+\sum_{i=1}^n\log\left(r_i^2\right)-\frac{1}{2\sigma^2}\sum_{i=1}^n r_i^2. \tag{1.5}$$

Taking derivation with respect to the parameter σ and equating to zero in equation (1.4) the MLE of σ is in the following form, $\hat{\sigma}=\sqrt{(3n)^{-1}\sum_{i=1}^n r_i^2}$.

Maxwell distribution was first developed by James Clerk *Maxwell* in 1860 in the field of kinetic energy of gases which was later on extended by Ludwig Boltzmann [1]. Hence the distribution is called *Maxwell*-Boltzmann distribution or simply *Maxwell* distribution of velocity according to the named after these two pioneers. In the recent decades this distribution is applied in lifetime modelling, chemistry as well as in statistical mechanics [2], [3], [4].

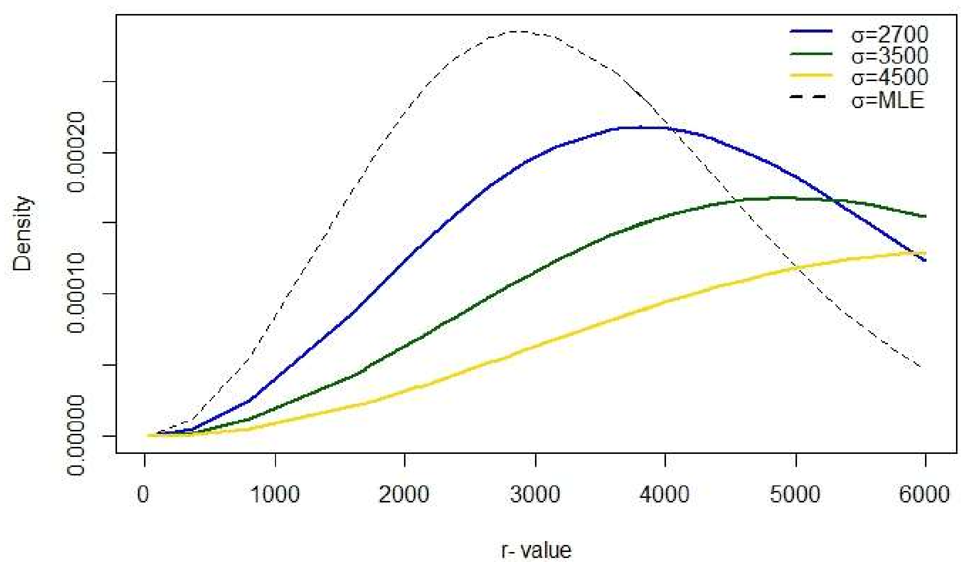


Figure 1.1: The PDF of *Maxwell* distribution for different value of parameter.

In the literature, we may find different statistical distributions that are used to model lifetime data. To know the behavior of failure time distribution of any product these distributions are applicable. Exponential distribution, Weibull distribution, gamma distribution, extreme value distribution etc. are very common statistical distributions used in lifetime modelling. But in practice, we may encounter situations in which the lifetime data obtained from experiment may not fit any of these popular distributions. Consequently, there is always an opportunity for other statistical distributions to be considered as a lifetime model. Thus, [5] and [6] first studied *Maxwell* distribution as a possible lifetime distribution model.

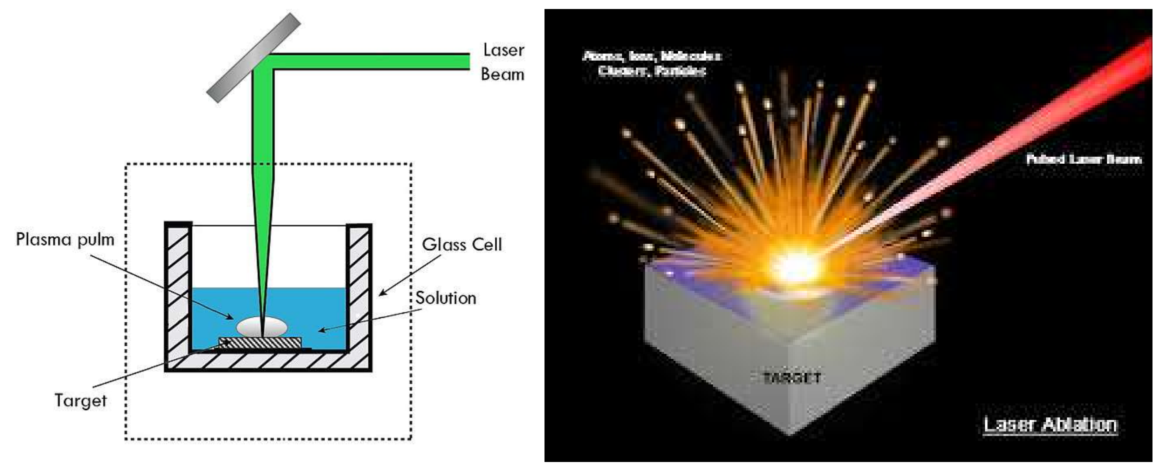


Figure 1.2: Laser Ablation in solution medium and pulsed laser beam **[7]**, **[8]**.

Laser ablation is a process of removing materials from a solid surface by irradiating it with laser beam. In laser ablation, discarded molecules emitted from a solid surface have streams with certain velocities. These molecules are shown to follow the *Maxwell* distribution [9].

In Chemistry research, *Maxwell* distribution diagrams are very appealing tools to chemist in these days [8]. Airbags are used to help the passenger reduce their speed in crash without getting injured. Airbags contain a mixture of different gases. During an accident, these gases undergo a chemical reaction and produce a new and harmless gas, nitrogen, which fills the airbag and saves passengers from extremely dangerous accident. From the kinetic theory of gases, it is shown that temperature is related to the average speed of molecules. Hence there must be a distribution of speeds for the airbag gas. It is shown in the literature airbag gases speed follow *Maxwell* distribution. With the increase of temperature, the number of molecules that are traveling at high speeds increases, and the speeds become more steadily distributed [9].

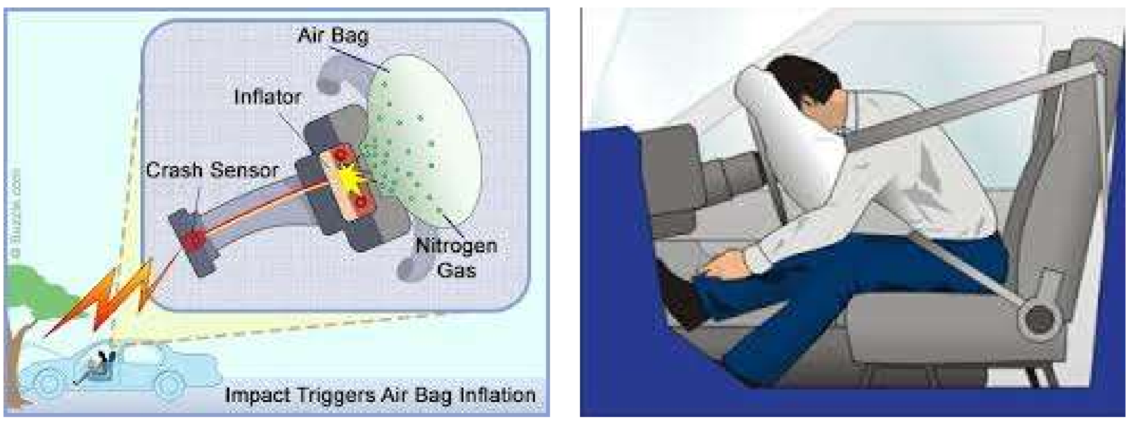


Figure 1.3: Impact triggers airbag inflation **[10]**, **[11]**.

Now, in statistics, a finite mixture distribution is a probability distribution that expressed as a linear combination of other probability distributions. This type of distribution is recommended to model a population that constitute with different subpopulations and the unknown weights represent the proportion of each subpopulation in the entire population. For example, in a vertical boring machine Manufacturing Company, the lifetimes of all the vertical boring machines produced by manufacturer may be considered as a population. However, these manufactured vertical boring machines may be divided into different subpopulations based on the particular machine or engineer who produced a particular boring machine.

On the other hand, infinite mixture distribution is a probability distribution when the components of mixing distributions are not countable. For example the t distribution with ν degrees of freedom is known to be a special infinite mixture of the normal distribution with inverse chi square distribution [12]. Now the question is, why do we need mixture distribution? Let us, for example, consider a distribution $f(r,\sigma)$ with parameter σ. When the parameter of the distribution σ is known, the parameter is fix value and thus it is a regular PDF of the distribution. Hence mixing distribution representing the parameter will not be needed. But in the situation where σ is unknown, we need to estimate the parameter. Thus the estimator has a distribution due to variability. To take this into account, we may mix the distribution of the estimate of the unknown parameter $h(\hat{\sigma})$ with the parent distribution $f(r,\hat{\sigma})$. Since the estimator may be continuous then we deal with infinite mixture distribution. Infinite mixture distribution can be defined as

$$f_R(r;\sigma) = \int f_{R|\Sigma}(r \mid \sigma) h_\Sigma(\sigma)\, d\sigma \tag{1.6}$$

Another reason, when we deal with finite mixture we deal only with mixing proportion not with mixing distribution. Recall the distribution $f(r,\sigma)$ with parameter σ. Then k components mixture of the distribution $f(r,\sigma)$ is the convex combination of the mixing proportion and the given distribution. Hence, finite mixture distribution can be defined as

$$F(r;\sigma) = \sum_{i=1}^{k} \theta_i F_i(r;\sigma) \tag{1.7}$$

where, $F_1(r), F_2(r), ..., F_k(r)$ are CDFs of the PDFs $f_1(r), f_2(r), ..., f_k(r)$, respectively, and θ_i are nonnegative quantities that sum to one; that is $0 \le \theta_i \le 1,\ (i = 1, 2, ..., k)$ and $\sum_{i=1}^{k} \theta_i = 1.$ Hence, the two components and three components mixture of distributions are respectively as

$$F(r) = \theta_1 F_1(r) + \theta_2 F_2(r);\ \text{where, } \theta_2 = (1 - \theta_1) \text{ and} \tag{1.8}$$

$$F(r) = \theta_1 F_1(r) + \theta_2 F_2(r) + \theta_3 F_3(r) \text{ where, } \theta_3 = (1 - \theta_1 - \theta_2). \tag{1.9}$$

1.2. Outlines of remaining chapters

In the following chapters we studied different mixture distributions such as *k* component finite *Maxwell* distribution, infinite *Maxwell* mixture distribution etc. along with various properties of the distributions. Also, a detail literature survey presented in Chapter 2.

Chapter 3 consists of preliminaries such as moment generating function, characteristic function, survival function, hazard function of mixture distribution in general. These properties for *k* component mixture distribution has also been discussed. Besides this, some special cases of finite *Maxwell* mixture distribution also been discussed. A short discussion on rescaled finite *Maxwell* mixture distribution introduced in the same chapter. In Chapter 4 similar properties has been discussed for infinite *Maxwell* distribution. In addition, we proposed a new probability density function which we called *tau square* distribution along with its properties.

The parameter estimation for both finite and infinite *Maxwell* mixture distribution has been argued in the Chapter 5. We used the expectation maximization (EM) algorithm for estimating MLE of parameters of finite *Maxwell* mixture distribution and method of moment is used for estimating parameters of infinite *Maxwell* distribution. As it is always recommended to provide application in any work, we have applied all the instrument which we obtained in the previous chapter in Chapter 6. Finally, Chapter 7 is for general summary and conclusion.

CHAPTER TWO

LITERATURE REVIEW

In the recent years both the finite and infinite mixture distributions become more popular among researcher than it was in before. Most of the researchers investigated different fundamental properties of the distributions, for example, derivation of PDF and CDF of the mixture distribution, different moments, moment generating function, characteristic function etc. Real life application of the mixture distribution has been provided by a few researchers. In the subsequent section a detail literature of infinite and finite mixture distributions is provided to get clearer concept about different properties of the distributions.

2.1. Literature review on finite mixture distribution

In the early age of mixture distribution researcher are interested on finite mixture distribution. Mixture distribution, particularly the finite mixture distribution was first used by [13] in modelling outliers. But [14] treat as the pioneer of the introduction of finite mixture distribution in statistical modelling. He introduced a two component mixture of *normal* distribution in an analysis of crab morphometry data. [15] considered a mixture of two distributions where the first component distribution is *exponentiated Pareto* and the second component distribution is *exponential* distributions. Whereas [16] studied mixture of two components from *exponentiated gamma* distributions. Both of the researcher demonstrated estimation of parameters using maximum likelihood and Bayes methods for complete and type II censored samples. They also discussed the reliability and hazard function under these sampling scheme.

Besides this [17] studied mode and median of mixture of two *inverse Weibull* distributions with different choice of parameters and graphs. The failure rate function's behaviour of the distribution has also been studied by the researchers. They estimated the parameters using EM algorithm. The performance of the estimation technique has been investigated using Monte Carlo simulation. [18] introduced mixture of *modified inverse Weibull* distribution. Their mixture is of two component mixture with different properties of the distribution such as reliability function, hazard function, different moments and moment generating function. They also investigate the relationship of their proposed mixture distribution with other two components finite mixture distributions. Finally, they provide a real life example in life time data of electronic components of the proposed distribution. The information matrix of two components finite mixture distribution has been studied by [19], [20] for mixture of two *normal* distributions and mixture of two *exponential* distributions, [21] for mixture of two *Pareto* distributions, [22] for two component *normal* and *Laplace* mixtures.

[23] considered a threefold mixture distribution where the component distributions are *lognormal*, *gamma* and *Weibull* distributions. In this study they investigated classical problems in mixture distributions such as identifiability, parameter estimation using EM algorithm and asymptotic properties of the estimators. They provided a practical example of their proposed model using length of hospital stay data from the University Virgen Macarena of Sevilla Hospital (Spain). [24] discussed the two component finite mixture of *normal* distribution where they demonstrated the test of homogeneity of dispersion

parameter using EM algorithm and at the end of the work they provide two genetic example to illustrate the application of EM test.

L-moment method of estimation is provided by for two components finite mixture of *Weibull* distribution. They introduced the method and compared the results with MLE method using simulation study. They found that their proposed method is better than MLE in terms of bias, mean absolute error, mean total and completion time of simulation algorithm. Also provide two real life example of Fatigue lives dataset and failure times for oral irrigators dataset. [25] introduced m-fold *Weibull* mixture distribution with application in reliability approximation of a system. [26] obtained MLE of the parameters of two components finite mixture of *normal* distribution and two component finite mixture of *lognormal* distributions using EM algorithm under right censoring. To validate the results also simulation study has been provided. In obtaining MLE of parameters of two components finite mixture distribution the EM algorithm technique also been implemented by [27], [28], [29], [30] and so on.

2.2. Literature review on infinite mixture distribution

In the literature we can find different kinds of infinite mixture distributions. [31], [32] and [33] defined and derived *chi-square* mixture of *chi-square*, *gamma* and *erlang* distributions with some basic properties and showed that all the mixtures are positively skewed and leptokurtic. Similarly, [34] and [35] studied different characteristics of *gamma* and *erlang* mixture of *normal* distributions and validate the properties by using simulation. *Rayleigh* mixture of *Rayleigh* and some other sampling distributions such as *chi-square, t* and *F* distributions were studied by [36] where they focused mainly on estimation of parameters and derivation of moments, characteristic function and shape characteristics. [37] studied *chi-square* mixture of *transformed gamma* distributions. None of the researcher provided practical application of their work in the real life. But some of them delivered simulation study only. On the other hand [38] showed an application in airborne communication transceiver and [38] showed application in coronary heart disease pattern detection of their investigated results.

2.3. Literature review on *Maxwell* mixture distribution

In the recent years, mixture of *Maxwell* distribution under type I censoring become more popular among the researchers. [39] and [40] studied *Maxwell* mixture distribution under type-I censoring. They estimated MLE of the parameters and their corresponding variance matrix. They also compared MLE with Bayes estimates under the square error loss function and precautionary loss function. [41] compared MLE and Bayes estimators of *Maxwell* distributions. In 2012, Krishna and Malik studied *Maxwell* distribution for life time distribution model. They, [42], estimated reliability function of the distribution with progressively Type-II censored data. But the pioneers of applying this distribution in the life time model were [43] and [44] in 1989.

2.4. Research questions

Based on the literature given in the previous section we may consider following problems

***Question 1*: Extension to *k* component *Maxwell* mixture distribution**

The literature has investigations of a mixture of two *Maxwell* subpopulations. What if we consider more than two subpopulations such as three, four up to k subpopulations based on the practical situation in the real life? Hence, we would like to extend the work to k subpopulations.

***Question 2*: Behavior of *Maxwell* distribution for complete sample**

Mixture distributions particularly *Maxwell* mixtures distributions has been studied/modeled in the literature for two components or subpopulations under type I censoring. But the behavior of the distribution under complete data has not yet been studied. Hence, we would like to study the properties of the model for complete sample.

***Question 3*: Reliability estimation in real life scenario**

As an application in the field of engineering or medical science, we would like to investigate the reliability of any given machine or survival of an individual after a certain amount of time. However, the problem is complicated by the fact that the parent population has a distribution that is composed of distinct subpopulations that can best be modeled by a k finite mixture *Maxwell* distribution.

CHAPTER THREE

FINITE MIXTURE OF MAXWELL DISTRIBUTION

In this chapter we will introduce k component mixture of *Maxwell* distribution. We will also demonstrate different properties of this mixture distribution such as moment generating function, characteristic function, survival function, hazard function. Some preliminaries of the distribution are given in the following sections.

3.1. Preliminaries of finite mixture distribution

In the Chapter one, finite mixture distribution has been defined. Here we introduce different properties of finite mixture distribution in general.

3.1.1. Moment generating function of finite mixture distribution

Theorem 3.1 The moment generating function of finite mixture distribution can be presented as weighted MGF of the component distribution.

Proof. Let R be a random variable having PDF $f(r)$. The moment generating function of R is defined by

$$M_R(t) = E\left[e^{tR}\right] = \int_{-\infty}^{\infty} e^{tr} f(r)\,dr, \tag{3.1}$$

provided $E\left[e^{tr}\right]$ exists for all values in the limit $-h < t < h,\ h > 0$. If $f(r)$ is a k component mixture distribution with mixing proportion θ, as defined in Chapter one equation (1.7), the equation (3.1) then can be written as

$$M_R(t) = E\left[e^{tR}\right] = \int_{-\infty}^{\infty} e^{tr} \sum_{i=1}^{k} \theta_i f_i(r) dr. \tag{3.2}$$

The above equation can be written as

$$
\begin{aligned}
M_R(t) &= \int_{-\infty}^{\infty} e^{tr} \sum_{i=1}^{k} \theta_i f_i(r) dr \\
&= \int_{-\infty}^{\infty} e^{tr} \left[\theta_1 f_1(r) + \theta_2 f_2(r) + \ldots + \theta_k f_k(r) \right] dr \\
&= \theta_1 \int_{-\infty}^{\infty} e^{tr} f_1(r) dr + \theta_2 \int_{-\infty}^{\infty} e^{tr} f_2(r) dr + \ldots + \theta_k \int_{-\infty}^{\infty} e^{tr} f_k(r) dr \\
&= \theta_1 M_{1R}(t) + \theta_2 M_{2R}(t) + \ldots + \theta_k M_{kR}(t).
\end{aligned}
$$

So, $M_R(t) = \sum_{i=1}^{k} \theta_i M_{iR}(t).$ (3.3)

Hence, we can say that MGF of finite mixture distribution is a weighted MGF of the component distribution. □

3.1.2. Characteristic function of finite mixture distribution

When we consider the moment generating function it is observed that moments of a distribution can be obtained from its MGF. But in certain cases for a particular distribution all the moments may not be found out from MGF. In addition, for some distributions MGF does not exist. To avoid such issues characteristic function may be used to find the moments and the probability function of the distribution. The characteristic function of the distribution of R is defined as

$$C_R(t) = E\left[e^{itR}\right] = \int_{-\infty}^{\infty} e^{itr} f(r) dr, \tag{3.4}$$

where $-h < t < h$, $i^2 = -1$, i is the imaginary number, and $h > 0$. Using characteristic function, the probability density function of a random variable R can be obtained as

$$f(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itr} C_R(t) dt. \tag{3.5}$$

This is called Levy's theorem or inverse Mellin transformation formula [45].

Theorem 3.2 The characteristic function of finite mixture distribution can be expressed as the weighted characteristic function of the component distribution.

Proof. Using the definition given in equation (3.4) the characteristic function for mixture distribution should be as the form as given underneath.

$$
\begin{aligned}
C_R(t) &= \int_{-\infty}^{\infty} e^{itr} \sum_{i=1}^{k} \theta_i f_i(r) dr \\
&= \int_{-\infty}^{\infty} e^{itr} \left[\theta_1 f_1(r) + \theta_2 f_2(r) + \ldots + \theta_k f_k(r)\right] dr \\
&= \theta_1 \int_{-\infty}^{\infty} e^{itr} f_1(r)\, dr + \theta_2 \int_{-\infty}^{\infty} e^{itr} f_2(r)\, dr + \ldots + \theta_k \int_{-\infty}^{\infty} e^{itr} f_k(r)\, dr \\
&= \theta_1 C_{1R}(t) + \theta_2 C_{2R}(t) + \ldots + \theta_k C_{kR}(t).
\end{aligned}
$$

So, $C_R(t) = \sum_{i=1}^{k} \theta_i C_{iR}(t).$ (3.6)

The above equation is sufficient for prove the theorem. □

Theorem 3.3 The Mellin transformation formula for finite mixture distribution can be presented as

$$f(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itr} \sum_{l=1}^{k} \theta_l C_{lR}(t)\, dt;\; l = 1, 2, \ldots, k \tag{3.7}$$

Proof. Due to equation (3.6), Mellin transformation formula, given in the equation (3.5), for mixture distribution can be presented as the form given in equation (3.7). □

3.1.3. Survival function of finite mixture distribution

Survival function is defined as the probability that an individual lives longer than time t. It is symbolized by $S(t)$. Hence, $S(t) = P$ [An individual stay alive more than time t] = $P[T > t] = \int_{t}^{\infty} f(r)\, dr$, where, T is survival time. From the definition of CDF $F(t)$ of T, we have, $S(t) = 1 - P$ [An individual dies before time t]$= 1 - F(t)$. Here $S(t)$ is a non-increasing function of time t such that $S(t) = 1$ for $t = 0$ and $S(t) = 0$ for $t = \infty$. That is, the probability that an individual will survive at time zero is at least 1 and will survive up to infinite time is zero. The function $S(t)$ is also known as the cumulative survival rate. The graph of $S(t)$ is called the survival curve. So, $S(t)$ is a monotonic decreasing continuous function. In terms of mixture distribution this function can be defined as

$$S(t) = 1 - \sum_{i=1}^{k} \theta_i F_i(t). \tag{3.8}$$

3.1.4. Hazard function of finite mixture distribution

The hazard function of survival time T provides the conditional failure rate. It is denoted by $h(t)$ and defined in terms of mixture distribution as

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{S(t)} = \frac{\sum_{i=1}^{k} \theta_i f_i(t)}{1 - \sum_{i=1}^{k} \theta_i F_i(t)}, \tag{3.9}$$

where, $F_i(t) =$ cumulative distribution function of *i*th component distribution, $f_i(t) =$ probability density function of *i*th component distribution, and $h(t) = 0$ for $t = 0$ and $h(t) = 1$ for $t = \infty$. The hazard function is also known as the instantaneous failure rate, force of mortality, conditional mortality rate, and age-specific failure rate.

3.1.5. Moments of finite mixture distribution

The *m*-th raw moment is defined as

$$\mu'_m = E\left(R^m\right) = \int_{-\infty}^{\infty} r^m f(r)\, dr\,. \tag{3.10}$$

Theorem 3.4 The *m*-th raw moment of finite mixture distribution can be presented as the weighted raw moment of component distribution.

Proof. In terms of mixture distribution equation (3.10) can be presented as

$$\begin{aligned} \mu'_m = E\left(R^m\right) &= \int_{-\infty}^{\infty} r^m \sum_{i=1}^{k} \theta_i f_i(r)\, dr \\ &= \sum_{i=1}^{k} \theta_i \int_{-\infty}^{\infty} r^m f_i(r)\, dr \\ &= \sum_{i=1}^{k} \theta_i E\left(R_i^m\right). \end{aligned}$$

So, $\mu'_m = \sum_{i=1}^{k} \theta_i \mu'_{im}$. (3.11)

3.2. Properties of finite mixture of *Maxwell* distribution

Consider a *Maxwell* distribution with PDF and CDF are given in the equation (1.1) and (1.2) respectively in Chapter one. Hence the mixture of *k Maxwell* distributions, according to the definition given in equation (1.7) in Chapter one, is presented by the PDF

$$f(r) = \sqrt{\frac{2}{\pi}} \frac{\theta_1}{\sigma_1^3} r^2 e^{-\frac{r^2}{2\sigma_1^2}} + \sqrt{\frac{2}{\pi}} \frac{\theta_2}{\sigma_2^3} r^2 e^{-\frac{r^2}{2\sigma_2^2}} + \ldots + \sqrt{\frac{2}{\pi}} \frac{\theta_k}{\sigma_k^3} r^2 e^{-\frac{r^2}{2\sigma_k^2}} \tag{3.12}$$

$$= \sum_{i=1}^{k} \sqrt{\frac{2}{\pi}} \frac{\theta_i}{\sigma_i^3} r^2 e^{-\frac{r^2}{2\sigma_i^2}}, \tag{3.13}$$

for $r > 0$, $\sigma_i > 0$, $0 \le \theta_i \le 1$, $(i = 1,2,\ldots,k)$, $\sum_{i=1}^{k} \theta_i = 1$ and assume without loss of generality that $\sigma_1 > \sigma_2 > \ldots > \sigma_k$. The following **Figure 3.1** shows the PDF of the distribution graphically. It shows that the *Maxwell* distribution is still right skewed distribution after mixing two or more components together. The figure is plotted using RStudio.Ink. The plot in upper left corner shows the PDF of two components *Maxwell* mixture distribution for equally proportion from each component for $\{\sigma_1, \sigma_2\}$= {2.5, 3.5}, {3.5, 4.5} and {5.3, 5.4}. One more component increases in the upper right plot. Where the mixing proportions are as $\{\theta_1, \theta_2, \theta_3\}$= {0.2, 0.3, 0.5} and scale parameters of the component distribution is given as $\{\sigma_1, \sigma_2, \sigma_3\}$= {2.5, 3.5, 4.5}, {3.5, 4.5, 5.5} and {5.3, 5.4, 5.5}. In similar fashion the lower right and lower left plots show the PDF of four components and five components mixture of *Maxwell* distribution for different mixing proportion and scale parameters.

And, the CDF of the above mentioned distribution is obtained using the equation (1.7) given in Chapter one as

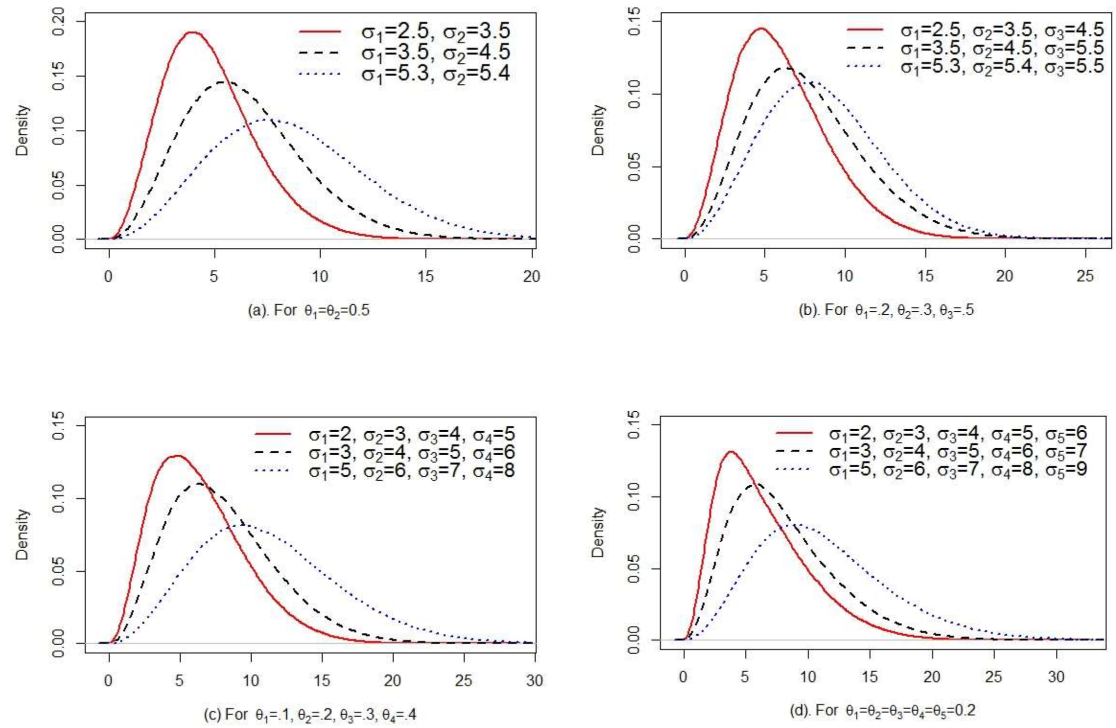


Figure 3.1: (a)-(d) PDF of finite mixture of *Maxwell* distributions for k = 2, 3, 4, and 5 at different mixing proportions and parameter values.

$$F(r)=\sum_{i=1}^{k}\theta_i\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{r^2}{2\sigma_i^2}\right). \tag{3.14}$$

Again, the following **Figure 3.2** demonstrates the CDF of the distribution. The plot in upper left corner shows the CDF of two components *Maxwell* mixture distribution for equally proportion from each component for $\{\sigma_1,\sigma_2\}$= {2.5, 3.5}, {3.5, 4.5} and {5.3, 5.4}. One more component increases in the upper right plot. Where the mixing proportions are as $\{\theta_1,\theta_2,\theta_3\}$= {0.2, 0.3, 0.5} and scale parameters of the component distribution are given as $\{\sigma_1,\sigma_2,\sigma_3\}$= {2.5, 3.5, 4.5}, {3.5, 4.5, 5.5} and {5.3, 5.4, 5.5}. In similar fashion the lower right and lower left plots show the CDF of four components and five components mixture of *Maxwell* distribution for different mixing proportions and scale parameters.

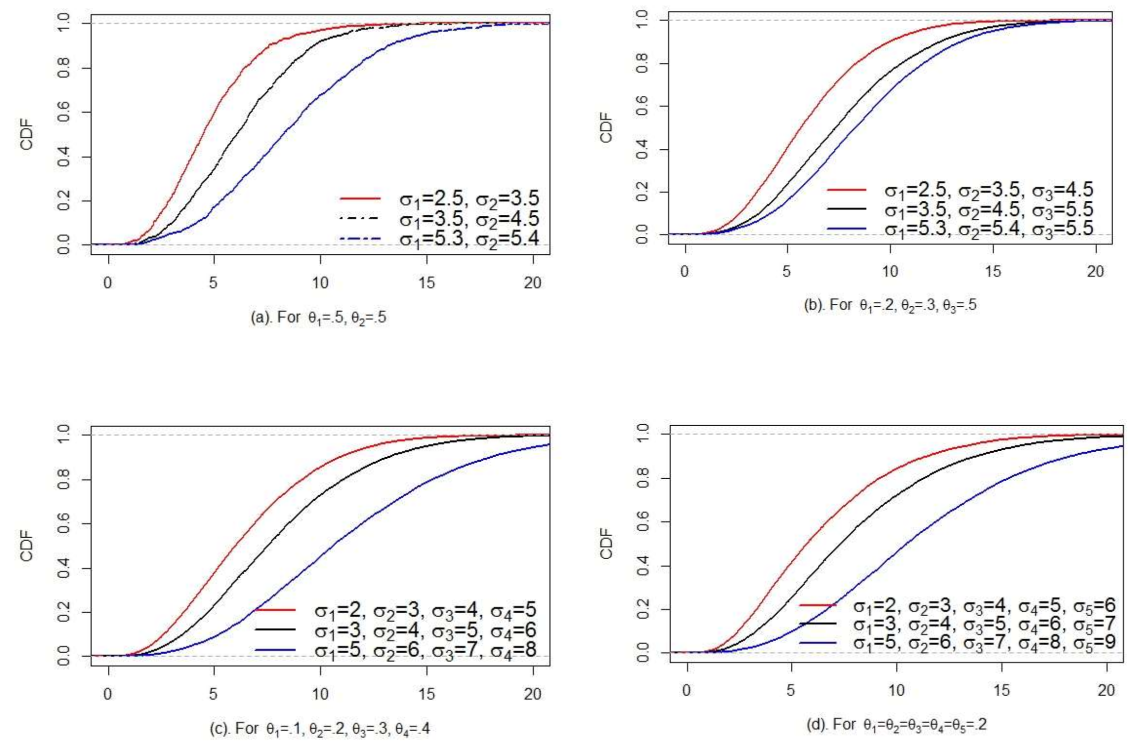


Figure 3.2 : (a)-(d) CDF of finite mixture of *Maxwell* distributions for k = 2, 3, 4 and 5 at different mixing proportions and parameter values**.**

Theorem 3.5 The integration of the equation (3.13) over the entire range is unity. Hence it is a PDF.

Proof. Taking integration in equation (3.13) we get,

$$\int_0^\infty f(r)\,dr = \sum_{i=1}^k \sqrt{\frac{2}{\pi}}\frac{\theta_i}{\alpha_i^3} I\left(\sigma_i^2\right), \tag{3.15}$$

where $I\left(\sigma_i^s\right) = \int_0^\infty \left(r^s e^{-\frac{r^2}{2\sigma_i^2}} \right) dr;\ i = 1,2,...,k$ and, $s = 2.$ Let, $\frac{r^2}{2\sigma_i^2} = p \Rightarrow r = \sigma_i\sqrt{2p}$

and taking differentiation on both sides we get, $dr = \sigma_i \frac{dp}{\sqrt{2p}}$. Hence,

$$\begin{aligned} I\left(\sigma_i^s\right) &= \int_0^\infty \left(\sigma_i\sqrt{2p}\right)^s e^{-p}\sigma_i \frac{dp}{\sqrt{2p}} \\ &= 2^{\left(\frac{s-1}{2}\right)} \sigma_i^{s+1} \int_0^\infty p^{\frac{s+1}{2}-1} e^{-p}\,dp. \end{aligned}$$

$$\text{So, } I\left(\sigma_i^s\right)=2^{\left(\frac{s-1}{2}\right)}\sigma_i^{s+1}\Gamma\left(\frac{s+1}{2}\right);\ i=1,2,\ldots,k\ . \tag{3.16}$$

Now putting s=2,we obtain $I\left(\sigma_i^2\right)=\sqrt{2}\sigma_i^3\Gamma\left(3/2\right)=\sigma_i^3\sqrt{\frac{\pi}{2}}.$, Therefore(3.15) can be written as $\int_0^\infty f\left(r\right)dr=\sum_{i=1}^k\theta_i=1$. Thus, the function given in (3.13) is a PDF. □

3.2.1. MGF of finite mixture of *Maxwell* distribution

Theorem 3.6 The moment generating function for finite mixture of *Maxwell* distribution is

$$M_R\left(t\right)=\sum_{i=1}^k\theta_i\left\{\sqrt{\frac{2}{\pi}}\sigma_i t+2\left(1+\sigma_i^2t^2\right)e^{\frac{1}{2}\sigma_i^2t^2}\Phi\left(\sigma_i t\right)\right\} \tag{3.17}$$

Here, $\Phi\left(.\right)$ is for CDF of standard normal distribution defined as $\Phi\left(z\right)=\int_{-\infty}^z\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt.$

Proof. Using the definition given in equation (3.1) we have

$$M_R\left(t\right)=E\left[e^{tR}\right]=\int_0^\infty e^{tr}\sum_{i=1}^k\sqrt{\frac{2}{\pi}}\frac{\theta_i}{\sigma_i^3}r^2e^{-\frac{r^2}{2\sigma_i^2}}dr;\quad i=1,2,\ldots,k$$

$$=\sqrt{\frac{2}{\pi}}\sum_{i=1}^k\frac{\theta_i}{\sigma_i^3}\int_0^\infty r^2e^{tr-\frac{r^2}{2\sigma_i^2}}dr\ .$$

$$\text{So, } M_R\left(t\right)=\sqrt{\frac{2}{\pi}}\sum_{i=1}^k\frac{\theta_i}{\sigma_i^3}I, \tag{3.18}$$

where $I=\int_0^\infty r^2e^{rt-\frac{r^2}{2\sigma_i^2}}dr;\quad i=1,2,\ldots,k$.

This can be rewritten as

$$I=e^{\frac{1}{2}\left(\sigma_i^2t^2\right)}\int_0^\infty r^2e^{-\frac{1}{2}\left(\frac{r-\sigma_i^2t}{\sigma_i}\right)^2}dr; \tag{3.19}$$

Now, let $\frac{r-\sigma_i^2t}{\sigma_i}=z$ which implies that $r=\sigma_iz+\sigma_i^2t$ and taking differentiation on both sides we get, $dr=\sigma_idz$.

Hence, equation (3.19) gets in to the form as

$$I = e^{\frac{1}{2}\left(\sigma_i^2 t^2\right)} \int_{-\sigma_i t}^{\infty} \left(\sigma_i z + \sigma_i^2 t\right)^2 e^{-\frac{z^2}{2}} \sigma_i dz$$
$$= e^{\frac{1}{2}\left(\sigma_i^2 t^2\right)} \sigma_i^3 \int_{-\sigma_i t}^{\infty} \left(z^2 + 2\sigma_i z t + \sigma_i^2 t^2\right) e^{-\frac{z^2}{2}} dz$$
$$= e^{\frac{1}{2}\left(\sigma_i^2 t^2\right)} \sigma_i^3 \left(\int_{-\sigma_i t}^{\infty} z^2 e^{-\frac{z^2}{2}} dz + \int_{-\sigma_i t}^{\infty} 2\sigma_i t z e^{-\frac{z^2}{2}} dz + \int_{-\sigma_i t}^{\infty} \sigma_i^2 t^2 e^{-\frac{z^2}{2}} dz \right)$$

Now considering that, Z is a standard normal variate with mean 0 and variance 1, and that CDF relationship of the stander normal distribution we have,

$$\int_{-\sigma_i t}^{\infty} z^2 e^{-\frac{z^2}{2}} dz = -\sigma_i t e^{-\frac{(\sigma_i t)^2}{2}} + \sqrt{2\pi}\Phi\left(\sigma_i t\right)$$
$$\int_{-\sigma_i t}^{\infty} z e^{-\frac{z^2}{2}} dz = e^{-\frac{(\sigma_i t)^2}{2}}$$
$$\int_{-\sigma_i t}^{\infty} e^{-\frac{z^2}{2}} dz = \sqrt{2\pi}\Phi\left(\sigma_i t\right)$$

Finally, by substitute this results in the equation (3.18) we attain the MGF of finite mixture of *Maxwell* distribution. □

Corollary 3.1 The MGF of finite mixture of *Maxwell* distribution is a weighted MGF of the component distributions.

Proof: The MGF of i-th component is [46]

$$M_{iR}\left(t\right) = \sqrt{\frac{2}{\pi}}\sigma_i t + 2\left(1 + \sigma_i^2 t^2\right) e^{\frac{1}{2}\left(\sigma_i^2 t^2\right)} \Phi\left(\sigma_i t\right)$$

Hence, equation (3.17) has the following expression

$$M_R\left(t\right) = \sum_{i=1}^{k} \theta_i M_{iR}\left(t\right)$$

Hence, prove. □

3.2.2. Characteristic function of finite mixture of *Maxwell* distribution

Theorem 3.7 The characteristic function for finite mixture of *Maxwell* distribution is

$$C_R\left(t\right) = \sum_{j=1}^{k} \theta_j \left\{ \sqrt{\frac{2}{\pi}}\sigma_j i t + 2\left(1 + \sigma_j^2 i^2 t^2\right) e^{\frac{1}{2}\sigma_j^2 i^2 t^2} \Phi\left(\sigma_j i t\right) \right\}. \tag{3.20}$$

Here, $\Phi(.)$ is for CDF of standard normal distribution defined as $\Phi(z)=\int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt.$

Proof. From equation (3.6)

$$C_R(t)=\int_0^{\infty}e^{itr}\sum_{j=1}^{k}\sqrt{\frac{2}{\pi}}\frac{\theta_j}{\sigma_j^3}r^2e^{-\frac{r^2}{2\sigma_j^2}}dr; \qquad j=1,2,...,k.$$

Here we use subscript j instead of subscript i since in characteristic function there is a i (iota) which may make confusion. Hence to avoid confusion we use subscript j.

$$C_R(t)=\sqrt{\frac{2}{\pi}}\sum_{j=1}^{k}\frac{\theta_j}{\sigma_j^3}\int_0^{\infty}r^2e^{itr-\frac{r^2}{2\sigma_j^2}}dr.$$

Hence, $$C_R(t)=\sqrt{\frac{2}{\pi}}\sum_{j=1}^{k}\frac{\theta_j}{\sigma_j^3}I, \tag{3.21}$$

where $I=\int_0^{\infty}r^2e^{itr-\frac{r^2}{2\sigma_j^2}}dr$ which can be simplified as $I=e^{\frac{1}{2}\sigma_j^2i^2t^2}\int_0^{\infty}r^2e^{-\frac{1}{2}\left(\frac{r-\sigma_j^2it}{\sigma_j}\right)^2}dr.$

Now, let $\frac{r-\sigma_i^2it}{\sigma_i}=z$ which implies that $r=\sigma_iz+\sigma_i^2it$ and taking differentiation with respect to r on both sides we get, $dr=\sigma_jdz$.

Hence, the integration can be written in the form

$$\begin{aligned}I&=e^{\frac{1}{2}\sigma_j^2i^2t^2}\int_{-\sigma_jit}^{\infty}\left(\sigma_jz+\sigma_j^2it\right)^2e^{-\frac{z^2}{2}}\sigma_jdz\\&=e^{\frac{1}{2}\sigma_j^2i^2t^2}\sigma_j^3\int_{-\sigma_jit}^{\infty}\left(z^2+2\sigma_jzit+\sigma_j^2i^2t^2\right)e^{-\frac{z^2}{2}}dz\\&=e^{\frac{1}{2}\sigma_j^2i^2t^2}\sigma_j^3\left(\int_{-\sigma_jit}^{\infty}z^2e^{-\frac{z^2}{2}}dz+\int_{-\sigma_jit}^{\infty}2\sigma_jitze^{-\frac{z^2}{2}}dz+\int_{-\sigma_jit}^{\infty}\sigma_i^2i^2t^2e^{-\frac{z^2}{2}}dz\right)\end{aligned}$$

Now considering that, Z is a standard normal variate with mean 0 and variance 1, and that CDF relationship of the stander normal distribution we have,

$$\int_{-\sigma_jit}^{\infty}z^2e^{-\frac{z^2}{2}}dz=-\sigma_jte^{-\frac{(\sigma_jit)^2}{2}}+\sqrt{2\pi}\Phi\left(\sigma_jit\right)$$

$$\int_{-\sigma_jit}^{\infty}ze^{-\frac{z^2}{2}}dz=e^{-\frac{(\sigma_jit)^2}{2}}$$

$$\int_{-\sigma_j it}^{\infty} e^{-\frac{z^2}{2}} dz = \sqrt{2\pi}\Phi\left(\sigma_j it\right)$$

Finally, after substituting equation (3.21) gives the results.

Hence, the desired result is obtained. □

Corollary 3.2 The characteristics function of finite mixture of *Maxwell* distribution is a weighted characteristic function of the component distributions.

Proof. Same as proof give in corollary 3.1. □

3.2.3. Survival function and hazard function of finite mixture of *Maxwell* distribution

Theorem 3.8 Survival function of finite mixture of *Maxwell* distribution is given as

$$S\left(t\right) = 1 - \sum_{i=1}^{k} \theta_i \frac{2}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{t^2}{2\sigma_i^2}\right). \tag{3.22}$$

Proof. Using definition given in equation (3.8) and the CDF given in equation (3.14) we get the result given in equation (3.22). □

Theorem 3.9 Hazard function of finite mixture of *Maxwell* distribution is given as

$$h\left(t\right) = \sum_{i=1}^{k} \frac{\theta_i}{\sigma_i^3} t^2 e^{-\frac{t^2}{2\sigma_i^2}} \bigg/ \left\{\sqrt{\frac{\pi}{2}} - \sum_{i=1}^{k} \theta_i \gamma\left(\frac{3}{2}, \frac{t^2}{2\sigma_i^2}\right)\right\}. \tag{3.23}$$

Proof. Using definition given in equation (3.9) we have from equation (3.13) and (3.22)

$$h\left(t\right) = \frac{\sum_{i=1}^{k} \sqrt{\frac{2}{\pi}} \frac{\theta_i}{\sigma_i^3} t^2 e^{-\frac{t^2}{2\sigma_i^2}}}{1 - \sum_{i=1}^{k} \theta_i \frac{2}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{t^2}{2\sigma_i^2}\right)}$$

$$= \frac{\sqrt{\frac{2}{\pi}} \sum_{i=1}^{k} \frac{\theta_i}{\sigma_i^3} t^2 e^{-\frac{t^2}{2\sigma_i^2}}}{\frac{2}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2} - \sum_{i=1}^{k} \theta_i \gamma\left(\frac{3}{2}, \frac{t^2}{2\sigma_i^2}\right)\right]}$$

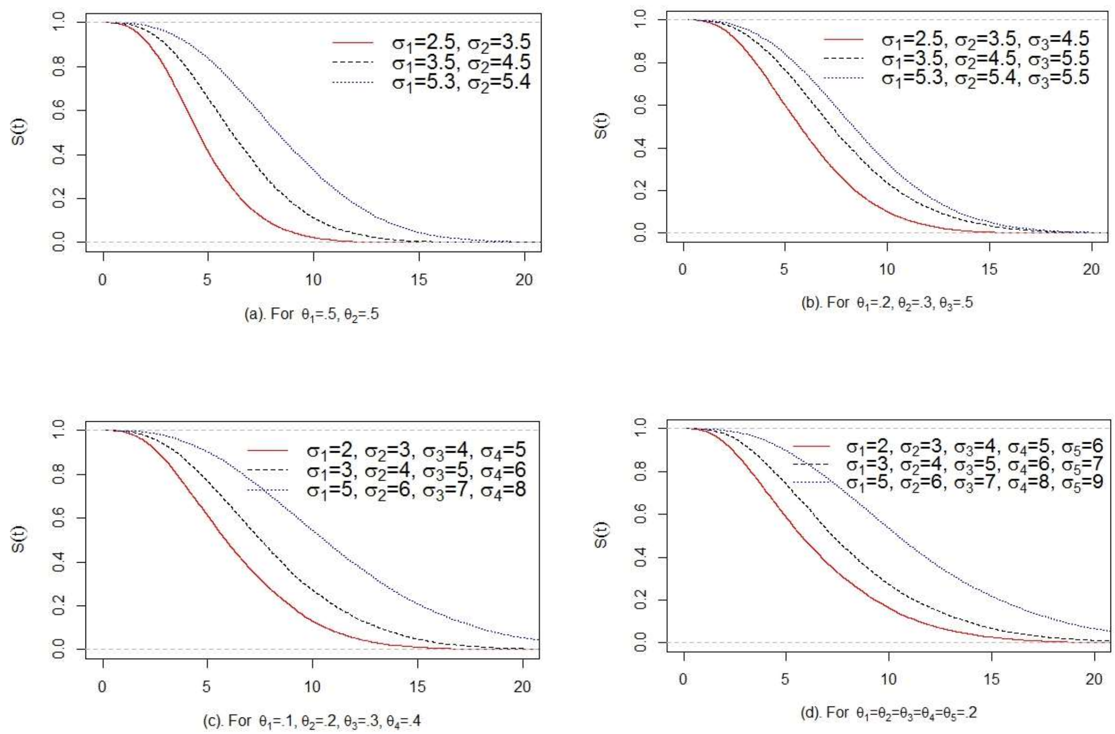


Figure 3.3: (a)-(b) Survival function of finite mixture of *Maxwell* distribution for k=2,3,4, and 5 at different mixing proportions and parameter values.

$$= \frac{\sum_{i=1}^{k} \frac{\theta_i}{\sigma_i^3} t^2 e^{-\frac{t^2}{2\sigma_i^2}}}{\left[\frac{\sqrt{\pi}}{\sqrt{2}} - \sum_{i=1}^{k} \theta_i \gamma\left(\frac{3}{2}, \frac{t^2}{2\sigma_i^2}\right)\right]}$$

$$= \frac{\sum_{i=1}^{k} \frac{\theta_i}{\sigma_i^3} t^2 e^{-\frac{t^2}{2\sigma_i^2}}}{\left[\sqrt{\frac{\pi}{2}} - \sum_{i=1}^{k} \theta_i \gamma\left(\frac{3}{2}, \frac{t^2}{2\sigma_i^2}\right)\right]}.$$

Hence, the theorem is proved. □

Theorem 3.10 The m th moments of finite mixture of *Maxwell* distribution is given as

$$\mu'_m = \frac{2^{\frac{m+2}{2}}}{\sqrt{\pi}} \Gamma\left(\frac{m+3}{2}\right) \sum_{i=1}^{k} \theta_i \sigma_i^m. \tag{3.24}$$

Proof. Recall equation (3.11) and (3.13) to obtain the mth raw moment of finite mixture of *Maxwell* distribution as

$$\mu'_m = \sqrt{\frac{2}{\pi}} \sum_{i=1}^{k} \frac{\theta_i}{\sigma_i^3} \int_0^\infty r^{m+2} e^{-\frac{r^2}{2\sigma_i^2}} dr. \tag{3.25}$$

Now let $\frac{r^2}{2\sigma_i^2} = p$ which implies that $r = \sigma_i \sqrt{2p}$. Taking differentiation on both sides with respect to r we get, $\frac{r}{\sigma_i^2} = \frac{dp}{dr}$. After putting the value of r this can be written as $dr = \frac{\sigma_i dp}{\sqrt{2p}}$.

Hence,

$$\mu'_m = \frac{2^{\frac{m+2}{2}}}{\sqrt{\pi}} \sum_{i=1}^{k} \theta_i \sigma_i^m \int_0^\infty p^{\frac{m+1}{2}} e^{-p} dp$$

$$= \frac{2^{\frac{m+2}{2}}}{\sqrt{\pi}} \sum_{i=1}^{k} \theta_i \sigma_i^m \Gamma\left(\frac{m+3}{2}\right).$$

Hence m-th raw moment of finite mixture of *Maxwell* distribution. □

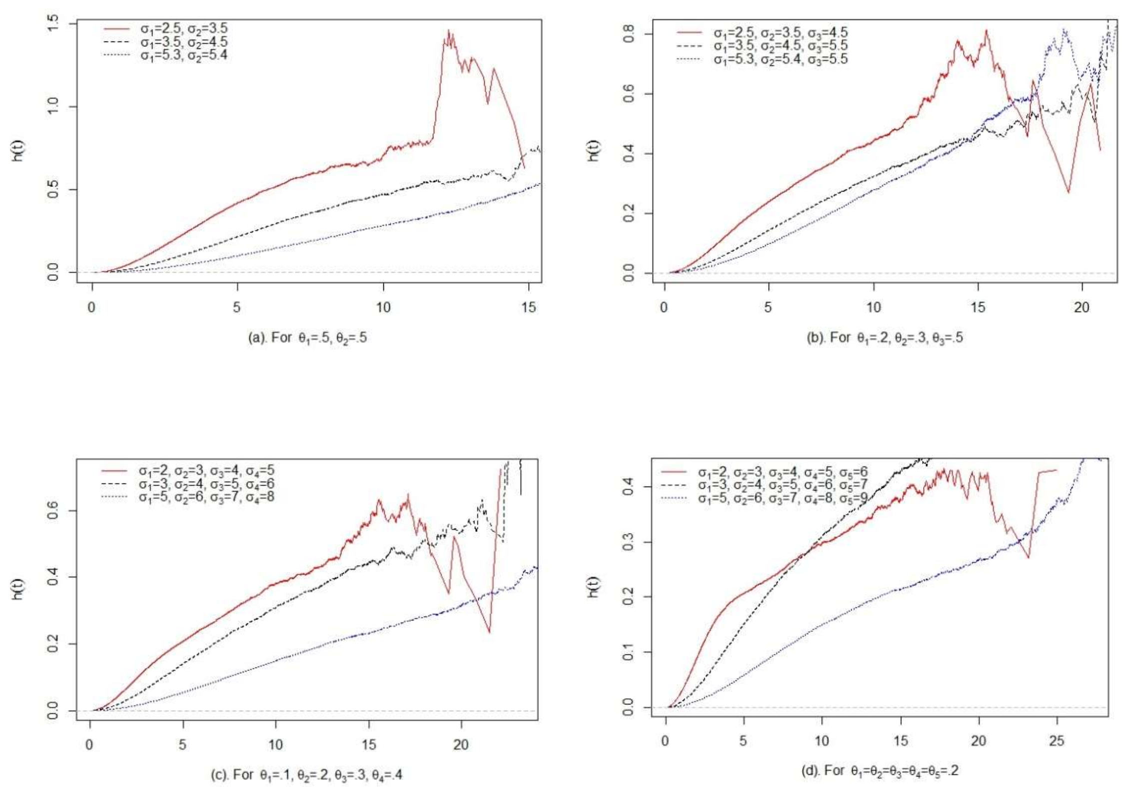


Figure 3.4: (a)-(d) Hazard function of finite mixture of *Maxwell* distribution for k=2,3,4, and 5 at different mixing proportions and parameter values.

Corollary 3.3 The first four moments of finite mixture of *Maxwell* distribution are

$$\mu_1' = 2\sqrt{2/\pi}\sum_{i=1}^{k}\theta_i\sigma_i \ ,$$
$$\mu_2' = 3\sum_{i=1}^{k}\theta_i\sigma_i^2 \ ,$$
$$\mu_3' = 8\sqrt{2/\pi}\sum_{i=1}^{k}\theta_i\sigma_i^3 \ , \text{ and}$$
$$\mu_4' = 15\sum_{i=1}^{k}\theta_i\sigma_i^4 .$$

Proof. By substituting m=1, 2, 3 and 4 the first four raw moments finite mixture of *Maxwell* distribution can be obtaining. □

3.3. Some special *k* component mixture of *Maxwell* distribution

In this part we discuss some special cases of finite mixture of *Maxwell* distribution. Particularly we consider

1. When θ_i is the probability from a discrete *uniform* distribution
2. When θ_i is the probability from a *binomial* distribution

3.3.1. When θ_i is the probability from a discrete *uniform* distribution

When the mixing parameter θ_i is a probability from discrete *uniform* distribution, then it can be presented as follows

$$\theta_i = P\left(X = i\right) = \frac{1}{k};\ i = 1,2,...,k. \tag{3.26}$$

Hence, the PDF given in (3.13) can be written as

$$f\left(r\right) = \sqrt{\frac{2}{\pi}}\frac{1}{k}\sum_{i=1}^{k}\frac{1}{\sigma_i^3}r^2 e^{-\frac{r^2}{2\sigma_i^2}} . \tag{3.27}$$

Therefore, finite mixture of *Maxwell* distribution in this case is nothing but the average of the component Maxwell distribution. Also we can say that, in case of mixing proportion following discrete uniform distribution then the proportion is equally distributed among the component distributions. That is, if we mix 2 components then mixing proportion should be 0.5 and 0.5, if we mix 3 components then the proportion should be 0.33, 0.33 and 0.33, likewise, for 4 components mixture the proportion should be 0.25, 0.25,0.25 and 0.25 and so on.

3.3.2. When θ_i is the probability from a *binomial* distribution

In a situation when θ_i is a probability from a *binomial* distribution, then θ_i can be defined as

$$\theta_i = P(X = i-1) = \binom{k-1}{i-1} p^{(i-1)} (1-p)^{k-i};\ i = 1, 2, \ldots, k. \tag{3.28}$$

Consequently, the k component mixture of *Maxwell* distribution can be written as

$$f(r) = \sqrt{\frac{2}{\pi}} \sum_{i=1}^{k} \binom{k-1}{i-1} p^{(i-1)} (1-p)^{k-i} \frac{r^2}{\sigma_i^3} e^{-\frac{r^2}{2\sigma_i^2}}; \tag{3.29}$$

for, r>0, σ>0. Suppose that, in a manufacturing company different types of ball pens have been produced. If a ball pen has been taken randomly either it is defective or good. At the same time the pen may be any of different types. Say the k types of ball pen has been produced in the company. Then the probability of getting a particular type of pen which is good or bad can be modeled using the equation given in (3.29).

3.4. Summary of the finite mixture of *Maxwell* distribution

In this chapter, we developed finite mixture distribution in general. We studied different properties of the distribution. We implemented these properties for a particular distribution name k component mixture of *Maxwell* distribution. The mentionable properties of the distribution are MGF, CF, survival function and hazard function. We saw that the MGF of mixture of *Maxwell* distribution is the weighted MGF of component distributions. Similar to MGF, the CF of mixture distribution is also presented as weighted CF of component distributions. As moments are very important to study the shape characteristics of a distribution, we derived first four moments of the finite mixture of *Maxwell* distribution. We considered some special cases of the proposed distribution which may be implemented in some real life scenario. In the next chapter we will discuss different properties of infinite mixture distribution. Then we will implement these properties for infinite mixture of *Maxwell* distribution.

CHAPTER FOUR

INFINITE MIXTURE OF MAXWELL DISTRIBUTION

In this chapter we introduce infinite mixture of *Maxwell* distribution. Also in introducing this mixture, we proposed a new *chi square* like distribution which we named as *tau square* distribution. Like in the previous chapter, we will also discuss different properties of the distribution. Moment generating function, characteristic function, survival function, hazard function and moments are the key properties of the distribution which discussed here. The probability density function along with the distribution function are also presented graphically.

4.1. Preliminaries of infinite mixture distribution

Before starting with properties of infinite mixture distribution let us recall the PDF of infinite mixture distribution defined in equation (1.6) as $f_R\left(r;\sigma\right)=\int f_{R|\Sigma}\left(r\mid\sigma\right)h_\Sigma\left(\sigma\right)d\sigma$.

From the definition it is clear that the mixture distribution is the product of conditional distribution of random variable given the parameter and the unconditional distribution of parameter to be estimated where the distribution of parameter should be integrated out.

In the subsequent sections we develop theorems based on this definition of mixture distribution. In this chapter we introduce different properties of infinite mixture distribution in general. Now we would like to make clear some terminologies of mixture distribution by introducing some definitions.

Definition 4.1 The unconditional distribution of parameter which is going to mix with conditional distribution of random variable given parameter and going to be integrated out is termed as ***mixing distribution.*** In equation (1.6) $h_{\Sigma}\left(\sigma\right)$ is the mixing distribution.

Definition 4.2 The conditional distribution of random variable *R* given the parameter σ which is going to mix with unconditional distribution of parameter is known as ***mixed distribution.*** In equation (1.6) $f_{R|\Sigma}\left(r \mid \sigma\right)$ is the mixed distribution.

Definition 4.3 The resulting unconditional distribution which is obtained by integrating the product of mixing and mixed distributions with respect to parameter is called ***mixture distribution.*** In equation (1.6) $f_{R}\left(r\right)$ is the mixture distribution.

Here we also would like to provide some mathematical preliminaries which are very helpful in mathematical derivations. For example,

The gamma function is defined as [47]

$$\Gamma\left(x\right)=\int_{0}^{\infty} t^{x-1}e^{-t}dt;\ \ For \quad \operatorname{Re} x>0. \tag{4.1}$$

This above function is sometimes known as Eulerian integral of second kind.

The lower incomplete gamma function is given as [48] 3.381(1)

$$\gamma\left(a,x\right)=\int_{0}^{x} t^{a-1}e^{-t}dt;\ \ For \ \operatorname{Re}\ a>0\,. \tag{4.2}$$

The power series expansion of the above equation is of the following form

$$\gamma\left(a,x\right)=\sum_{k=0}^{\infty}\frac{x^{a+k}e^{-x}\Gamma\left(a\right)}{\Gamma\left(a+k+1\right)};\ \ For\ \operatorname{Re}\ a>0\,. \tag{4.3}$$

Consider the following relationship

$$\gamma\left(a,x;b\right)=\int_{0}^{x} t^{a-1}e^{-t-bt^{-1}}dt\,. \tag{4.4}$$

This is lower generalized incomplete gamma function and an upper generalized incomplete gamma function is defined as [49], [50].

$$\Gamma(a,x;b)=\int_{x}^{\infty} t^{a-1}e^{-t-bt^{-1}}dt\ . \tag{4.5}$$

Whereas the generalized incomplete gamma function is defined as [49], [50].

$$\Gamma(a,0;b)=\int_{0}^{\infty} t^{a-1}e^{-t-bt^{-1}}dt\ =2b^{a/2}K_a\left(2\sqrt{b}\right). \tag{4.6}$$

Here $K_a\left(2\sqrt{b}\right)$ is McDonald function [49], [50]. A decomposition formula of generalized gamma function is also be possible as

$$\gamma(a,x;b)+\Gamma(a,x;b)=\Gamma(a,0;b). \tag{4.7}$$

A relationship between beta function and gamma function is given as following form [47]

$$\mathrm{B}(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{4.8}$$

4.1.1. Moment generating function of infinite mixture distribution

Theorem 4.1 The MGF of infinite mixture distribution is the expectation of the MGF of the **mixed** distribution.

Proof. Recall the MGF defined in Chapter three in equation (3.1). Hence, MGF for infinite mixture distribution can be written as,

$$\begin{aligned} M_R(t)=E\left[e^{tR}\right]&=\int_{-\infty}^{\infty} e^{tr} f_R(r)\,dr \\ &=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{tr} f_{R|\Sigma}(r\mid\sigma)h(\sigma)\,dr\,d\sigma \\ &=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{tr} f_{R|\Sigma}(r\mid\sigma)\,dr\right]h(\sigma)\,d\sigma \\ &=\int_{-\infty}^{\infty} E_{R|\Sigma}\left[e^{tR}\right]h(\sigma)\,d\sigma \\ &=\int_{-\infty}^{\infty} M_{R|\Sigma}(t)h(\sigma)\,d\sigma. \end{aligned} \tag{4.9}$$

So, $M_R(t)=E_{\Sigma}\left[M_{R|\Sigma}(t)\right]$. (4.10)

This completes the proof of the theorem. □

4.1.2. Characteristic function of infinite mixture distribution

Theorem 4.2 The characteristic function of infinite mixture distribution is the expectation of the characteristic function of **mixed** distribution.

Proof. Recall the characteristic function defined in Chapter three in equation (3.4). The characteristic function for infinite mixture of distribution can be written as

$$\begin{aligned}
C_R(t) = E\left[e^{itR}\right] &= \int_{-\infty}^{\infty} e^{itr} f_R(r)\,dr \\
&= \int_{-\infty}^{\infty} e^{itr} \int_{-\infty}^{\infty} f_{R|\Sigma}(r \mid \sigma)\, h(\sigma)\, d\sigma dr \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{itr} f_{R|\Sigma}(r \mid \sigma)\, dr \right] h(\sigma)\, d\sigma \\
&= \int_{-\infty}^{\infty} E_{R|\Sigma}\left[e^{itR}\right] h(\sigma)\, d\sigma \\
&= \int_{-\infty}^{\infty} C_{R|\Sigma}(t)\, h(\sigma)\, d\sigma.
\end{aligned}$$

$$\text{So, } C_R(t) = E_{\Sigma}\left[C_{R|\Sigma}(t)\right]. \tag{4.11}$$

Hence, this proves the theorem. □

4.1.3. Survival function and hazard function for infinite mixture distribution

Theorem 4.3 Survival and hazard function of infinite mixture distribution can be expressed in terms of expectation of mixed distribution.

Proof. The survival function for infinite mixture of distribution can be written as

$$\begin{aligned}
S(t) &= 1 - F(t) \\
&= 1 - \int_{-\infty}^{t} f_R(r)\, dr \\
&= 1 - \int_{-\infty}^{t} \int_{-\infty}^{\infty} f_{R|\Sigma}(r \mid \sigma)\, h(\sigma)\, d\sigma dr \\
&= 1 - \int_{-\infty}^{\infty} h(\sigma) \int_{-\infty}^{t} f_{R|\Sigma}(r \mid \sigma)\, dr d\sigma \\
&= 1 - \int_{-\infty}^{\infty} h(\sigma)\, F_{R|\Sigma}(t)\, d\sigma.
\end{aligned}$$

$$\text{So, } S(t) = 1 - E_{\Sigma}\left[F_{R|\Sigma}(t)\right]. \tag{4.12}$$

And hence, hazard function for infinite mixture distribution can be written as

$$h(t) = \frac{f_R(t)}{1-F(t)} = \frac{1}{1-E_\Sigma\left[F_{R|\Sigma}(t)\right]}\int_{-\infty}^{\infty} f(t\,|\,\sigma)h(\sigma)d\sigma. \tag{4.13}$$

Hence, this proves the theorem. . □

4.1.4. Moments of infinite mixture distribution

Theorem 4.4 The *m* th moment of infinite mixture distribution is the expectation of the *m* th raw moment of the mixed distribution.

Proof. The *m* th moment of infinite mixture distribution can be written as

$$\begin{aligned}\mu'_m = E\left(R^m\right) &= \int_{-\infty}^{\infty} r^m f_R(r)\,dr \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r^m f_{R|\Sigma}(r\,|\,\sigma)h(\sigma)\,d\sigma dr \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} r^m f_{R|\Sigma}(r\,|\,\sigma)\,dr\right]h(\sigma)\,d\sigma \\ &= \int_{-\infty}^{\infty}\left[E_{R|\Sigma}\left(R^m\right)\right]h(\sigma)\,d\sigma \\ &= E_\Sigma\left[E_{R|\Sigma}\left(R^m\right)\right].\end{aligned}$$

So, $\mu'_m = E_\Sigma\left[\mu'_m(R\,|\,\Sigma)\right].$ (4.14)

Hence, this proves the theorem. □

4.2. Developing infinite mixture of *Maxwell* distribution

For developing infinite mixture of *Maxwell* distribution we first develop a new *chi square* like distribution which we named as *tau square* distribution. This proposed distribution has properties which are very close to *chi square* distribution. Secondly we mix it with *Maxwell* distribution.

4.2.1. Formation of the *tau square* distribution.

Recall the probability density function of *Maxwell* distribution given in equation (1.1). The distribution has only scale parameter σ. The MLE of σ is $\hat{\sigma} = \sqrt{(3n)^{-1}\sum_{i=1}^{n} r_i^2}$, that is

$\hat{\sigma}^2 = (3n)^{-1}\sum_{i=1}^{n} r_i^2$ [51], and the distribution of $U = \frac{1}{2\sigma^2}\sum_{i=1}^{n} r_i^2$ is *gamma* $(3n/2,1)$.

Hence, $U = \frac{3n\hat{\sigma}^2}{2\sigma^2} \sim gamma(3n/2,1)$ [52]. Now let the transformation

$$\mathrm{T}^2 = \frac{2U}{3n} = \frac{\hat{\sigma}^2}{\sigma^2} \quad (\mathrm{T}^2 \text{ Read as } \textit{tau square}). \tag{4.15}$$

Hence the new random variable $\frac{3n}{2}\mathrm{T}^2 = U$ is *gamma* $\sim (3n/2,1)$. The density function of U is

$$f(u) = \frac{1}{\Gamma(3n/2)} u^{\frac{3n}{2}-1} e^{-u};\; u > 0. \tag{4.16}$$

With transformation Jacobian $J(u \to \tau^2) = \frac{du}{d\tau^2} = \frac{3n}{2}$ the density function of T^2 is

$$h(\tau^2) = f_U(\tau^2)|J|.$$

$$= \frac{1}{\Gamma(3n/2)}\left(\frac{3n}{2}\tau^2\right)^{\frac{3n}{2}-1} e^{-\frac{3n}{2}\tau^2}\left|\frac{3n}{2}\right|.$$

$$\text{So,}\quad h(\tau^2) = \frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}(\tau^2)^{\frac{3n}{2}-1} e^{-\frac{3n}{2}\tau^2};\; \tau^2 > 0,\; n \ge 1. \tag{4.17}$$

The only parameter n of the distribution is the sample size. The proof that the function given in equation (4.17) is a PDF is given in the following theorem.

Theorem 4.5 The *tau square* function given in equation (4.17) is PDF.

Proof. The conditions that a function $h(x)$ is a density function are

$$(i)\; h(x) \ge 0;\; \textit{for all } x \in (-\infty, \infty) \text{ and } (ii)\; \int_{-\infty}^{\infty} h(x) = 1.$$

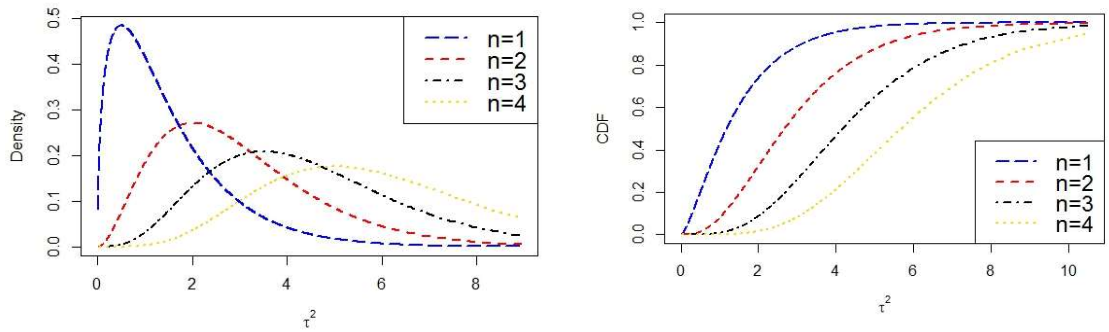


Figure 4.1: PDF and CDF of *tau square* distribution.

Since, $n > 0$, $\tau^2 > 0$ and any value of exponential function also greater than zero, hence it is obvious for the function given in equation (4.17) that $h\left(\tau^2\right) > 0$. As for the second condition, let us take integration on both side of the equation (4.17).

$$\int_0^\infty h\left(\tau^2\right)d\tau^2 = \frac{\left(3n/2\right)^{\frac{3n}{2}}}{\Gamma\left(3n/2\right)}\int_0^\infty \left(\tau^2\right)^{\frac{3n}{2}-1} e^{-\frac{3n}{2}\tau^2}\, d\tau^2 . \tag{4.18}$$

Let, $\frac{3n}{2}\tau^2 = p$ which implies that $\tau^2 = 2p/3n$ and taking differentiation on both sides with respect to τ^2, we get $d\tau^2 = \frac{2}{3n}dp$. Hence,(4.18) can be written as,

$$\int_0^\infty h\left(\tau^2\right)d\tau^2 = \frac{1}{\Gamma\left(3n/2\right)}\int_0^\infty p^{\frac{3n}{2}-1} e^{-p} dp .$$

After simplification, finally we get

$$\int_0^\infty h\left(\tau^2\right)d\tau^2 = \frac{1}{\Gamma\left(3n/2\right)}\Gamma\left(\frac{3n}{2}\right) = 1 .$$

Hence, the *tau square* function given in equation (4.17) is a probability density function.□

4.2.2. *Tau square* mixture of *Maxwell* distribution

As discussed earlier in Chapter one the reason why do we use mixture distribution. In case of known distribution parameter, we don’t bother with mixing distributions since still it is a regular distribution. But in a scenario when we deal with parameter estimation, the estimate of parameter may have its own distribution due to variability. To take this uncertainty into account we may mix the distribution of the estimate of the unknown parameter with the parent distribution.

Theorem 4.6 The probability density function of *tau square* mixture of *Maxwell* distribution is given as

$$f(r;\sigma)=\sqrt{\frac{3n}{\pi}}\frac{(3n/2)}{\Gamma(3n/2)}\frac{r^2}{\sigma^3}G(0),\ r>0, \tag{4.19}$$

where $G(k)=\Gamma\left(\frac{3}{2}(n-1)-k,0;\frac{3nr^2}{4\sigma^2}\right)$ and $\Gamma(a,0;b)$ is generalized *gamma* function given in equation (4.6).

Proof. Let the PDF of *Maxwell* distribution, given in equation (1.1), can be reparametrized using the transformation given in (4.15) as

$$f(r\,|\,\tau)=\sqrt{\frac{2}{\pi}}\frac{1}{\hat{\sigma}^3}r^2e^{-\left(\frac{r^2}{2\hat{\sigma}^2}\right)}. \tag{4.20}$$

For fixed observed value of $\hat{\sigma}$, the random variable R may be distributed as a *Maxwell* distribution. After simplifying the above equation (4.20) we get,

$$f(r;\sigma\,|\,\tau)=\sqrt{\frac{2}{\pi}}\frac{1}{\sigma^3\tau^3}r^2e^{-\left(\frac{r^2}{2\sigma^2\tau^2}\right)};\quad r>0. \tag{4.21}$$

Therefore, the *tau square* mixture of *Maxwell* distribution, using the definition given in equation (1.6) in Chapter one, is given by

$$\begin{aligned}
f(r;\sigma)&=\int_0^\infty f(r\,|\,\tau)h(\tau^2)\,d\tau^2\\
&=\sqrt{\frac{2}{\pi}}r^2\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\int_0^\infty\frac{1}{\hat{\sigma}^3}e^{-\left(\frac{r^2}{2\hat{\sigma}^2}\right)}\left(\tau^2\right)^{\frac{3n}{2}-1}e^{-\frac{3n}{2}\tau^2}\,d\tau^2\\
&=\sqrt{\frac{2}{\pi}}r^2\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\int_0^\infty\frac{1}{\sigma^3\tau^3}e^{-\left(\frac{r^2}{2\sigma^2\tau^2}\right)}\left(\tau^2\right)^{\frac{3n}{2}-1}e^{-\frac{3n}{2}\tau^2}\,d\tau^2;\ \text{since},\ \mathrm{T}^2=\frac{\hat{\sigma}^2}{\sigma^2}.
\end{aligned}$$

$$\text{So, } f(r;\sigma)=\sqrt{\frac{2}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\frac{r^2}{\sigma^3}I(\tau^2);\quad r>0 \text{ and } \tau^2>0. \tag{4.22}$$

Where, $I(\tau^2)=\int_0^\infty (\tau^2)^{\frac{3n-5}{2}} e^{-\frac{1}{2\sigma^2}\left(\frac{r^2}{\tau^2}+3n\sigma^2\tau^2\right)}d\tau^2$. Another form of this integral can be presented on the light of Chaudhry Ahmed distribution [53]. The PDF and CDF of Chaudhry-Ahmed distribution are respectively as follows

$$f(x)=2\sqrt{\alpha/\pi}e^{-\left(\sqrt{\alpha}x-\sqrt{\beta}x^{-1}\right)^2},\quad \text{and} \tag{4.23}$$

$$F(x)=\frac{1}{\sqrt{\pi}}e^{2\sqrt{\alpha\beta}}\gamma\left(\frac{1}{2},\alpha x^2;\alpha\beta\right). \tag{4.24}$$

Now consider the integral part of the equation (4.22) and let $\frac{3n}{2}\tau^2=x$ and taking differentiation with respect to x we get $d\tau^2=\frac{2}{3n}dx$. Hence the integral can be represented as

$$\begin{aligned} I(\tau^2) &= \left(\frac{2}{3n}\right)^{\frac{3n-3}{2}}\int_0^\infty (x)^{\frac{3n-5}{2}} e^{-\left(\frac{3nr^2}{4\sigma^2 x}+x\right)}dx \\ &= \left(\frac{2}{3n}\right)^{\frac{3n-3}{2}}\int_0^\infty (x)^{\frac{3n-3}{2}-1} e^{-x-\frac{3nr^2}{4\sigma^2 x}}dx \\ &= \left(\frac{2}{3n}\right)^{\frac{3n-3}{2}}\Gamma\left(\frac{3}{2}(n-1),0;\frac{3nr^2}{4\sigma^2}\right). \end{aligned}$$

Consequently, equation (4.22) can be written as the form as given in theorem. Hence prove the theorem. □

Corollary 4.1 In terms of McDonald function equation (4.19) can be presented as

$$f(r;\sigma)=\sqrt{\frac{3n}{\pi}}\frac{3n}{\Gamma(3n/2)}\frac{r^2}{\sigma^3}\left(\frac{3nr^2}{4\sigma^2}\right)^{\frac{3(n-1)}{4}}K_{\frac{3}{2}(n-1)}\left(\frac{r}{\sigma}\sqrt{3n}\right). \tag{4.25}$$

Therefore any of the equations (4.19), (4.22), and (4.25) can be called *tau square* mixture of *Maxwell* distribution.

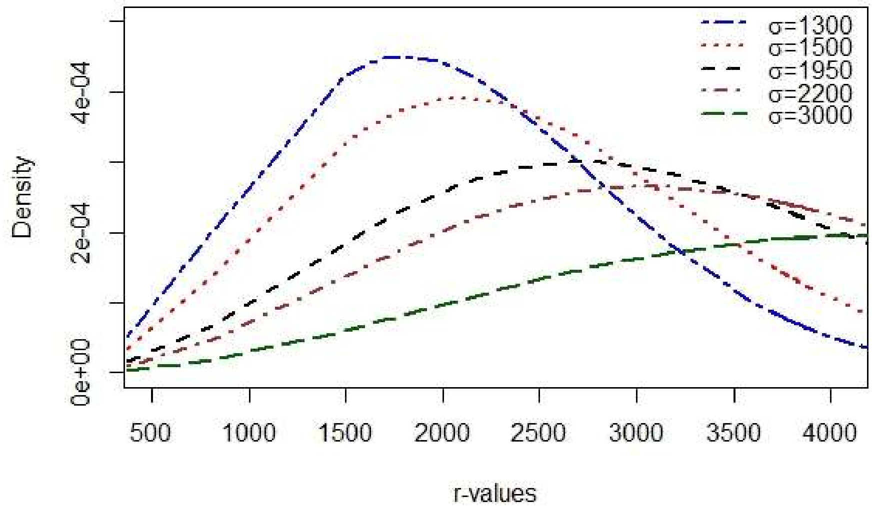


Figure 4.2: PDF comparison of *Tau square* mixture of *Maxwell* distribution.

Theorem 4.7 The density given in equation (4.22) integrates to unity over the entire range of r.

Proof. Taking integration on both sides of the equation (4.22) we get,

$$\int_0^\infty f(r;\sigma)dr = \sqrt{\frac{2}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\frac{1}{\sigma^3}\int_0^\infty \left(\tau^2\right)^{\frac{3n-5}{2}} e^{-\frac{1}{2}\left(3n\tau^2\right)}\int_0^\infty r^2 e^{-\frac{1}{2\sigma^2}\left(\frac{r^2}{\tau^2}\right)}drd\tau^2. \tag{4.26}$$

Let, $\frac{1}{2\sigma^2}\left(\frac{r^2}{\tau^2}\right) = p$ which implies that $r = \sigma\tau\sqrt{2p}$ and taking differentiation on both sides with respect to r we get $\frac{r}{\sigma^2\tau^2} = \frac{dp}{dr}$. By substituting the value of r finally we obtain $dr = \sigma\tau\frac{dp}{\sqrt{2p}}$. Hence, equation (4.26) can be written as

$$\int_0^\infty f(r;\sigma)dr = \sqrt{\frac{2}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\frac{1}{\sigma^3}\int_0^\infty \left(\tau^2\right)^{\frac{3n-5}{2}} e^{-\frac{1}{2}\left(3n\tau^2\right)}(\sigma\tau)^3\int_0^\infty \left(\sqrt{2p}\right)e^{-p}dpd\tau^2$$

$$= \frac{2}{\sqrt{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\int_0^\infty \left(\tau^2\right)^{\frac{3n-2}{2}} e^{-\frac{1}{2}\left(3n\tau^2\right)}\int_0^\infty p^{\frac{3}{2}-1}e^{-p}dpd\tau^2$$

$$= \frac{2}{\sqrt{\pi}} \frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)} \Gamma(3/2) \int_0^\infty (\tau^2)^{\frac{3n-2}{2}} e^{-\frac{1}{2}(3n\tau^2)} d\tau^2$$

$$= \frac{2}{\sqrt{\pi}} \frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)} \left(\frac{\sqrt{\pi}}{2}\right) \int_0^\infty (\tau^2)^{\frac{3n}{2}-1} e^{-\frac{1}{2}(3n\tau^2)} d\tau^2$$

$$= \int_0^\infty \frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)} (\tau^2)^{\frac{3n}{2}-1} e^{-\frac{3n}{2}\tau^2} d\tau^2.$$

Since, the integral part is the integration over the entire range of *tau square* distribution results to unity, hence prove the theorem. □

Theorem 4.8 The cumulative distribution function of *tau square* mixture of *Maxwell* distribution is

$$F(r;\sigma) = \sum_{k=0}^\infty \frac{G(k)}{\Gamma((5/2)+k)\Gamma(3n/2)} \left(\frac{3nr^2}{4\sigma^2}\right)^{\frac{3}{2}+k} ; \quad r > 0. \tag{4.27}$$

Where, $G(k) = \Gamma\left(\frac{3}{2}(n-1)-k, 0; \frac{3nr^2}{4\sigma^2}\right)$.

Proof. By definition,

$$F(r;\sigma) = \int_0^r f(r;\sigma)\, dr$$

$$= \int_0^r \sqrt{\frac{2}{\pi}} \frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)} \frac{r^2}{\sigma^3} \int_0^\infty (\tau^2)^{\frac{3n-5}{2}} e^{-\frac{1}{2\sigma^2}\left(\frac{r^2}{\tau^2}+3n\sigma^2\tau^2\right)} d\tau^2 dr.$$

$$\text{So, } F(r;\sigma) = \sqrt{\frac{2}{\pi}} \frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)} \frac{1}{\sigma^3} \int_0^\infty (\tau^2)^{\frac{3n-5}{2}} e^{-\frac{1}{2}(3n\tau^2)} \int_0^r r^2 e^{-\frac{1}{2\sigma^2}\left(\frac{r^2}{\tau^2}\right)} dr d\tau^2. \tag{4.28}$$

Now, Let, $\frac{1}{2\sigma^2}\left(\frac{r^2}{\tau^2}\right) = p$ which implies that $r = \sigma\tau\sqrt{2p}$ and taking differentiation on both sides with respect to *r* we get $\frac{r}{\sigma^2\tau^2} = \frac{dp}{dr}$. By substituting the value of *r* finally we obtain $dr = \sigma\tau \frac{dp}{\sqrt{2p}}$. Hence, equation (4.28) can be written as

$$F(r;\sigma)=\sqrt{\frac{2}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\frac{1}{\sigma^3}\int_0^\infty\left(\tau^2\right)^{\frac{3n-5}{2}}e^{-\frac{1}{2}\left(3n\tau^2\right)}\left(\sigma\tau\right)^3\int_0^{\frac{1}{2\sigma^2}\left(\frac{r^2}{\tau^2}\right)}\left(\sqrt{2p}\right)e^{-p}dpd\tau^2$$

$$=\frac{2}{\sqrt{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\int_0^\infty\left(\tau^2\right)^{\frac{3n-2}{2}}e^{-\frac{1}{2}\left(3n\tau^2\right)}\int_0^{\frac{1}{2\sigma^2}\left(\frac{r^2}{\tau^2}\right)}p^{\frac{3}{2}-1}e^{-p}dpd\tau^2.$$

Which simplifies to the following equation

$$F(r;\sigma)=\frac{2}{\sqrt{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\int_0^\infty\left(\tau^2\right)^{\frac{3n}{2}-1}e^{-\frac{1}{2}\left(3n\tau^2\right)}\gamma\left(\frac{3}{2},\frac{r^2}{2\sigma^2\tau^2}\right)d\tau^2$$

Now using the relationship given in equation (4.3) above can be written as

$$F(r;\sigma)=\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\sum_{k=0}^\infty\frac{I\left(\tau^2\right)}{\Gamma\left(\frac{3}{2}+k+1\right)}\left(\frac{r^2}{2\sigma^2}\right)^{\frac{3}{2}+k}. \tag{4.29}$$

Where, $I\left(\tau^2\right)=\int_0^\infty\left(\tau^2\right)^{\frac{3n-5}{2}-k}e^{-\frac{1}{2\sigma^2}\left(\frac{r^2}{\tau^2}+3n\sigma^2\tau^2\right)}d\tau^2$. Using the equation (4.6) this can be simplified to

$$I\left(\tau^2\right)=\left(\frac{2}{3n}\right)^{\frac{3n-3}{2}-k}\Gamma\left(\frac{3}{2}(n-1)-k,0;\frac{3nr^2}{4\alpha^2}\right).$$

By substituting this integral result to (4.29) and simplification equation (4.27) can be obtained. □

Corollary 4.2 Using McDonald function the CDF given in equation (4.27) can presented as

The form given below

$$F(r;\sigma)=\sum_{k=0}^\infty\frac{2}{\Gamma\left(\frac{5}{2}+k\right)\Gamma\left(\frac{3n}{2}\right)}\left(\frac{3nr^2}{4\sigma^2}\right)^{\frac{3}{4}(n+1)+\frac{k}{2}}K_{\frac{3}{2}(n-1)-k}\left(\frac{r}{\sigma}\sqrt{3n}\right). \tag{4.30}$$

Here, $K_a(.)$ is McDonald function.

4.3. Properties of infinite mixture of *Maxwell* distribution

In earlier section 4.1 we discussed different properties of infinite mixture distribution in general. In this section we implement these properties for *tau square* mixture of *Maxwell* distribution. All the results are given in a form of theorem.

4.3.1. Moment generating function of *tau square* mixture of *Maxwell* distribution

Theorem 4.9 The moment generating function of *tau square* mixture of *Maxwell* distribution is

$$M_R\left(t\right)=E_{\mathrm{T}^2}\left[\sqrt{\frac{2}{\pi}}\alpha\tau t+2\left(1+\alpha^2\tau^2t^2\right)e^{\frac{\alpha^2\tau^2t^2}{2}}\Phi\left(\alpha\tau t\right)\right]. \tag{4.31}$$

Here, $\Phi\left(.\right)$ is for CDF of standard normal distribution defined as $\Phi\left(z\right)=\int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt.$

Proof. By using the definition of MGF,

$$\begin{aligned}
M_R\left(t\right)&=E\left[e^{tR}\right]=\int_{-\infty}^{\infty}e^{tr}f\left(r\right)dr\\
&=\int_0^{\infty}e^{tr}\sqrt{\frac{2}{\pi}}\frac{\left(3n/2\right)^{\frac{3n}{2}}}{\Gamma\left(3n/2\right)}\frac{r^2}{\sigma^3}\int_0^{\infty}\left(\tau^2\right)^{\frac{3n-5}{2}}e^{-\frac{1}{2\sigma^2}\left(\frac{r^2}{\tau^2}+3n\sigma^2\tau^2\right)}d\tau^2dr\\
&=\sqrt{\frac{2}{\pi}}\frac{\left(3n/2\right)^{\frac{3n}{2}}}{\Gamma\left(3n/2\right)}\frac{1}{\sigma^3}\int_0^{\infty}\left(\tau^2\right)^{\frac{3n-5}{2}}\int_0^{\infty}r^2e^{-\frac{1}{2\sigma^2}\left(\frac{r^2}{\tau^2}+3n\sigma^2\tau^2-2\sigma^2tr\right)}drd\tau^2\\
&=\sqrt{\frac{2}{\pi}}\frac{\left(3n/2\right)^{\frac{3n}{2}}}{\Gamma\left(3n/2\right)}\frac{1}{\sigma^3}\int_0^{\infty}\left(\tau^2\right)^{\frac{3n-5}{2}}e^{-\frac{1}{2}\left(3n\tau^2\right)}\int_0^{\infty}r^2e^{-\frac{1}{2\sigma^2}\left(\left(\frac{r}{\tau}-\sigma^2\tau t\right)^2-\left(\sigma^2\tau t\right)^2\right)}drd\tau^2\\
&=\sqrt{\frac{2}{\pi}}\frac{\left(3n/2\right)^{\frac{3n}{2}}}{\Gamma\left(3n/2\right)}\frac{1}{\sigma^3}\int_0^{\infty}\left(\tau^2\right)^{\frac{3n-5}{2}}e^{-\frac{\tau^2}{2}\left(3n-\sigma^2t^2\right)}\int_0^{\infty}r^2e^{-\frac{1}{2}\left(\frac{r-\sigma^2\tau^2t}{\tau\sigma}\right)^2}drd\tau^2
\end{aligned}$$

Now, let $\frac{r-\sigma^2\tau^2t}{\tau\sigma}=z$ which implies that $r=z\tau\sigma+\sigma^2\tau^2t$ and taking differentiation on both sides with respect to *r* we get, $dr=\tau\sigma dz.$

Hence, $M_R(t)=\sqrt{\frac{2}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\frac{1}{\sigma^3}\int_0^\infty(\tau^2)^{\frac{3n}{2}-1}e^{-\frac{\tau^2}{2}(3n-\sigma^2t^2)}I(\tau^2)\,d\tau^2$, where

$$\begin{aligned}
I(\tau^2)&=\int_{-\alpha\tau t}^{\infty}(z\tau\sigma+\sigma^2\tau^2t)^2\,e^{-\frac{1}{2}z^2}\,\tau\sigma dz\\
&=\tau^3\sigma^3\int_{-\alpha\tau t}^{\infty}(z^2+2z\sigma\tau t+\sigma^2\tau^2t^2)e^{-\frac{1}{2}z^2}dz\\
&=\tau^3\sigma^3\left\{\int_{-\alpha\tau t}^{\infty}z^2e^{-\frac{1}{2}z^2}dz+2\sigma\tau t\int_{-\alpha\tau t}^{\infty}ze^{-\frac{1}{2}z^2}dz+\sigma^2\tau^2t^2\int_{-\alpha\tau t}^{\infty}e^{-\frac{1}{2}z^2}dz\right\}.
\end{aligned}$$

Consequently, the MGF can be presented as:

$$\begin{aligned}
M_R(t)=\sqrt{\frac{2}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\frac{1}{\sigma^3}\int_0^\infty(\tau^2)^{\frac{3n}{2}-1}e^{-\frac{\tau^2}{2}(3n-\sigma^2t^2)}\\
\times\tau^3\sigma^3\left[I_{\tau^2}(2)+2\sigma\tau tI_{\tau^2}(1)+\sigma^2\tau^2t^2I_{\tau^2}(0)\right]d\tau^2.
\end{aligned}\tag{4.32}$$

where,

$$I_{\tau^2}(2)=\int_{-\sigma\tau t}^{\infty}z^2e^{-\frac{1}{2}z^2}dz=-\sigma\tau te^{-\frac{\sigma^2\tau^2t^2}{2}}+\sqrt{2\pi}\Phi(\sigma\tau t).$$

$$I_{\tau^2}(1)=\int_{-\sigma\tau t}^{\infty}ze^{-\frac{1}{2}z^2}dz=e^{-\frac{\sigma^2\tau^2t^2}{2}}.$$

$$I_{\tau^2}(0)=\int_{-\sigma\tau t}^{\infty}e^{-\frac{1}{2}z^2}dz=\sqrt{2\pi}\Phi(\sigma\tau t).$$

Finally, putting these values in the equation (4.32) and simplification, the results given in equation (4.31) will attain. □

The moment generating function here derived for *tau square* mixture of *Maxwell* distribution is crucial as it generates different raw moments. These moments are very important in finding estimate of parameter using method of moment.

4.3.2. Characteristic function of *tau square* mixture of *Maxwell* distribution

Theorem 4.10 The characteristic function of *tau square* mixture of *Maxwell* distribution is

$$C_R(t) = E_{\Gamma^2}\left[\sqrt{\frac{2}{\pi}}\sigma\tau it + 2\left(1+\sigma^2\tau^2 i^2 t^2\right)e^{\frac{\sigma^2\tau^2 i^2 t^2}{2}}\Phi(\sigma\tau it)\right]. \tag{4.33}$$

Here, $\Phi(.)$ is for CDF of standard normal distribution defined as $\Phi(z) = \int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt.$

Proof. By definition,

$$\begin{aligned}
C_R(t) &= E\left[e^{itR}\right] = \int_{-\infty}^{\infty} e^{itr} f(r)\,dr \\
&= \int_0^{\infty} e^{itr}\sqrt{\frac{2}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\frac{r^2}{\sigma^3}\int_0^{\infty}\left(\tau^2\right)^{\frac{3n-5}{2}} e^{-\frac{1}{2\sigma^2}\left(\frac{r^2}{\tau^2}+3n\sigma^2\tau^2\right)}d\tau^2 dr \\
&= \sqrt{\frac{2}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\frac{1}{\sigma^3}\int_0^{\infty}\left(\tau^2\right)^{\frac{3n-5}{2}} e^{-\frac{1}{2}\left(3n\tau^2\right)}\left[\int_0^{\infty} r^2 e^{-\frac{1}{2\sigma^2}\left(\frac{r^2}{\tau^2}-2\sigma^2 itr\right)}dr\right]d\tau^2 \\
&= \sqrt{\frac{2}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\frac{1}{\sigma^3}\int_0^{\infty}\left(\tau^2\right)^{\frac{3n-5}{2}} e^{-\frac{1}{2}\left(3n\tau^2\right)}\left[\int_0^{\infty} r^2 e^{-\frac{1}{2\sigma^2}\left(\left(\frac{r}{\tau}-\sigma^2\tau it\right)^2-\left(\sigma^2\tau it\right)^2\right)}dr\right]d\tau^2 \\
&= \sqrt{\frac{2}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\frac{1}{\sigma^3}\int_0^{\infty}\left(\tau^2\right)^{\frac{3n-5}{2}} e^{-\frac{1}{2}\left(3n\tau^2\right)}e^{-\frac{1}{2}\sigma^2\tau^2 t^2}\left[\int_0^{\infty} r^2 e^{-\frac{1}{2\sigma^2}\left(\frac{r}{\tau}-\sigma^2\tau it\right)^2}dr\right]d\tau^2 \\
&= \sqrt{\frac{2}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\frac{1}{\sigma^3}\int_0^{\infty}\left(\tau^2\right)^{\frac{3n-5}{2}} e^{-\frac{1}{2}\left(3n\tau^2\right)}e^{-\frac{1}{2}\sigma^2\tau^2 t^2}\left[\int_0^{\infty} r^2 e^{-\frac{1}{2}\left(\frac{r-\sigma^2\tau^2 it}{\sigma\tau}\right)^2}dr\right]d\tau^2.
\end{aligned}$$

Now, let, $\frac{r-\sigma^2\tau^2 it}{\sigma\tau} = z$ which implies that $r = z\tau\sigma + \sigma^2\tau^2 it$ and differentiate both sides with respect to r we get, $dr = \tau\sigma dz$.

Hence, $C_R(t) = \sqrt{\frac{2}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\frac{1}{\sigma^3}\int_0^{\infty}\left(\tau^2\right)^{\frac{3n-2}{2}} e^{-\frac{\tau^2}{2}\left(3n-\sigma^2 i^2 t^2\right)} I\left(\tau^2\right)d\tau^2$, where

$$\begin{aligned}
I\left(\tau^2\right) &= \int_{-\alpha\tau it}^{\infty}\left(z\tau\sigma+\sigma^2\tau^2 it\right)^2 e^{-\frac{1}{2}z^2}\tau\sigma dz \\
&= \tau^3\sigma^3\int_{-\alpha\tau t}^{\infty}\left(z^2+2z\sigma\tau it+\sigma^2\tau^2 i^2 t^2\right)e^{-\frac{1}{2}z^2}dz \\
&= \tau^3\sigma^3\left\{\int_{-\alpha\tau it}^{\infty} z^2 e^{-\frac{1}{2}z^2}dz + 2\sigma\tau it\int_{-\alpha\tau t}^{\infty} z e^{-\frac{1}{2}z^2}dz + \sigma^2\tau^2 i^2 t^2\int_{-\alpha\tau t}^{\infty} e^{-\frac{1}{2}z^2}dz\right\}.
\end{aligned}$$

Consequently, the CF can be presented as:

$$C_R(t)=\sqrt{\frac{2}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\frac{1}{\sigma^3}\int_0^{\infty}\left(\tau^2\right)^{\frac{3n}{2}-1}e^{-\frac{\tau^2}{2}\left(3n-\sigma^2 i^2 t^2\right)}$$
$$\times\tau^3\sigma^3\left[I_{\tau^2}(2)+2\sigma\tau itI_{\tau^2}(1)+\sigma^2\tau^2 i^2 t^2 I_{\tau^2}(0)\right]d\tau^2,. \qquad (4.34)$$

where,

$$I_{\tau^2}(2)=\int_{-\sigma\tau it}^{\infty} z^2 e^{-\frac{1}{2}z^2}dz=-\sigma\tau ite^{-\frac{\sigma^2\tau^2 i^2 t^2}{2}}+\sqrt{2\pi}\Phi(\sigma\tau it).$$
$$I_{\tau^2}(1)=\int_{-\sigma\tau it}^{\infty} ze^{-\frac{1}{2}z^2}dz=e^{-\frac{\sigma^2\tau^2 i^2 t^2}{2}}.$$
$$I_{\tau^2}(0)=\int_{-\sigma\tau it}^{\infty} e^{-\frac{1}{2}z^2}dz=\sqrt{2\pi}\Phi(\sigma\tau it).$$

Finally, using there results in the equation (4.34) and simplification we get the characteristic function given in equation (4.33). □

Like MGF, the characteristic function is also important for generation moments. The most important feature of this function is that, the characteristic function exists for all distribution whereas MGF may not exist for all. Moreover, to obtain the PDF of a distribution using Mellin transformation, the characteristic function is used.

4.3.3. Survival function of *tau square* mixture of *Maxwell* distribution

Theorem 4.11 The survival function of *tau square* mixture of *Maxwell* distribution is

$$S(r)=1-\sum_{k=0}^{\infty}\frac{G(k)}{\Gamma((5/2)+k)\Gamma(3n/2)}\left(\frac{3nr^2}{4\sigma^2}\right)^{\frac{3}{2}+k}. \qquad (4.35)$$

Where, $G(k)=\Gamma\left(\frac{3}{2}(n-1)-k,0;\frac{3nr^2}{4\sigma^2}\right)$.

Proof. Recall the definition of survival function given in Chapter three $S(r)=1-F(r)$ and put the value of CDF given equation (4.27) on that. The result is the equation given in equation (4.35). □

Once we have the survival function of *tau square* mixture of *Maxwell* distribution, we may use this in reliability estimation in engineering and survival estimation in medical statistics.

Theorem 4.12 The survival function of *tau square* mixture of *Maxwell* distribution can be presented as the form given in the equation underneath

$$S(r) = 1 - E_{\mathrm{T}}\left[F_{R|\mathrm{T}}(r)\right]. \tag{4.36}$$

where, R is for time and T is the random variable follows *tau square* distribution.

Proof. Recall the survival function of *tau square* mixture of *Maxwell* distribution. Then,

$$\begin{aligned}
S(r) &= 1 - \frac{2}{\sqrt{\pi}} \frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)} \int_0^\infty (\tau^2)^{\frac{3n}{2}-1} e^{-\frac{3n}{2}\tau^2} \gamma\left(\frac{3}{2}, \frac{r^2}{2\sigma^2\tau^2}\right) d\tau^2 \\
&= 1 - \int_0^\infty \left\{ \frac{2}{\sqrt{\pi}} \gamma\left(\frac{3}{2}, \frac{r^2}{2\sigma^2\tau^2}\right) \right\} \frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)} (\tau^2)^{\frac{3n}{2}-1} e^{-\frac{3n}{2}\tau^2} d\tau^2 \\
&= 1 - \int_0^\infty F_{R|\mathrm{T}}(r) \frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)} (\tau^2)^{\frac{3n}{2}-1} e^{-\frac{3n}{2}\tau^2} d\tau^2 \\
&= 1 - \int_0^\infty F_{R|\mathrm{T}}(r) h(\tau^2) d\tau^2 .
\end{aligned}$$

Here, $h(\tau^2)$ is the PDF of *tau square* distribution. This proves the theorem □

Therefore, the survival function of *tau square* mixture of *Maxwell* distribution is 1 minus the expected value of the CDF of conditional *Maxwell* distribution under unconditional *tau square* distribution. That is, once we have the CDF of mixed distribution then the survival function of mixture distribution can be easy attain form the expected value of mixing distribution.

4.3.4. Hazard function of *tau square* mixture of *Maxwell* distribution

Theorem 4.13 The hazard function of *tau square* mixture of *Maxwell* distribution is

$$h(t)=\frac{\sqrt{\frac{3n}{\pi}}\frac{3n}{2}\frac{t^2}{\sigma^3}G(0)}{\left[\Gamma\left(\frac{3n}{2}\right)-\sum_{k=0}^{\infty}\frac{G(k)}{\Gamma((5/2)+k)}\left(\frac{3nt^2}{4\sigma^2}\right)^{\frac{3}{2}+k}\right]}, \tag{4.37}$$

where $G(k)=\Gamma\left(\frac{3}{2}(n-1)-k,0;\frac{3nt^2}{4\sigma^2}\right)$.

Proof. Using definition of hazard function, we have,

$$h(t)=\frac{f(t)}{1-F(t)}=\frac{f(t)}{S(t)}.$$

By substituting values from equation (4.19) and (4.35) in the above equation and by simplifying the equation (4.37) can be attained. □

4.3.5. Moments of *tau square* mixture of *Maxwell* distribution

Theorem 4.14 The m-th raw moment of *tau square* mixture of *Maxwell* distribution is

$$\mu_m'=\frac{2^{(m+1)}}{\sqrt{\pi}\Gamma(3n/2)}\left(\frac{\sigma}{\sqrt{3n}}\right)^m\Gamma\left(\frac{m+3}{2}\right)\Gamma\left(\frac{3n+m}{2}\right). \tag{4.38}$$

Proof. By definition we have,

$$\begin{aligned}\mu_m'&=E(R^m)=\int_{-\infty}^{\infty}r^m f(r)\,dr\\&=\int_0^\infty r^m\sqrt{\frac{2}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\frac{r^2}{\sigma^3}\int_0^\infty(\tau^2)^{\frac{3n-5}{2}}e^{-\frac{1}{2\sigma^2}\left(\frac{r^2}{\tau^2}+3n\sigma^2\tau^2\right)}d\tau^2dr\\&=\sqrt{\frac{2}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\frac{1}{\sigma^3}\int_0^\infty(\tau^2)^{\frac{3n-5}{2}}e^{-\frac{3n}{2}(\tau^2)}\int_0^\infty r^{m+2}e^{-\frac{1}{2\sigma^2}\left(\frac{r^2}{\tau^2}\right)}drd\tau^2.\end{aligned}$$

Let, $\frac{1}{2\sigma^2}\left(\frac{r^2}{\tau^2}\right)=p$ which implies that $r=\sqrt{2p}\sigma\tau$ and taking differentiation on both sides with respect to r we get, $dr=\sigma^2\tau^2\frac{dp}{\sqrt{2p}\sigma\tau}$. Hence,

$$\mu_m'=\sqrt{\frac{2}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\frac{1}{\sigma^3}\int_0^\infty(\tau^2)^{\frac{3n-5}{2}}e^{-\frac{3n}{2}(\tau^2)}\int_0^\infty\left(\sqrt{2p}\sigma\tau\right)^{m+1}e^{-p}\sigma^2\tau^2dpd\tau^2$$

$$=\sqrt{\frac{2}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\frac{1}{\sigma^3}\int_0^\infty (\tau^2)^{\frac{3n-5}{2}} e^{-\frac{3n}{2}(\tau^2)} \left(\sqrt{2}\right)^{m+1} (\sigma\tau)^{m+3} \int_0^\infty (p)^{\frac{m+3}{2}-1} e^{-p} dp d\tau^2$$

$$=\sqrt{\frac{1}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\left(\sqrt{2}\right)^{m+2} (\sigma)^m \Gamma\left(\frac{m+3}{2}\right)\int_0^\infty (\tau^2)^{\frac{3n+m-2}{2}} e^{-\frac{3n}{2}(\tau^2)} d\tau^2$$

$$=\sqrt{\frac{1}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\left(\sqrt{2}\right)^{m+2} (\sigma)^m \Gamma\left(\frac{m+3}{2}\right)\int_0^\infty (\tau^2)^{\frac{3n+m}{2}-1} e^{-\frac{3n}{2}(\tau^2)} d\tau^2.$$

Again, let, $\frac{3n}{2}(\tau^2) = q \Rightarrow \tau^2 = \frac{2q}{3n}$ and taking differentiation on both sides with respect to τ^2 we get, $d\tau^2 = \frac{2dq}{3n}$. Therefore,

$$\mu'_m = \sqrt{\frac{1}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\left(\sqrt{2}\right)^{m+2} (\sigma)^m \Gamma\left(\frac{m+3}{2}\right)\int_0^\infty \left(\frac{2q}{3n}\right)^{\frac{3n+m}{2}-1} e^{-q}\frac{2dq}{3n}$$

$$=\sqrt{\frac{1}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\left(\sqrt{2}\right)^{m+2} (\sigma)^m \Gamma\left(\frac{m+3}{2}\right)\left(\frac{2}{3n}\right)^{\frac{3n+m}{2}} \int_0^\infty (q)^{\frac{3n+m}{2}-1} e^{-q} dq$$

$$=\sqrt{\frac{1}{\pi}}\frac{(3n/2)^{\frac{3n}{2}}}{\Gamma(3n/2)}\left(\sqrt{2}\right)^{m+2} (\sigma)^m \Gamma\left(\frac{m+3}{2}\right)\left(\frac{2}{3n}\right)^{\frac{3n+m}{2}} \Gamma\left(\frac{3n+m}{2}\right).$$

Finally, by simplifying we get the result given in the equation (4.38). □

Theorem 4.15 The first four raw moments of *tau square* mixture of *Maxwell* distribution is

$$\mu'_1 = \frac{4\sigma}{\sqrt{3n}\mathrm{B}(3n/2, 1/2)},$$
$$\mu'_2 = 3\sigma^2,$$
$$\mu'_3 = \frac{16\sigma^3(3n+1)}{\sqrt{3n}\mathrm{B}(3n/2, 1/2)}, \text{ and}$$
$$\mu'_4 = 15\sigma^4\left(1+\frac{2}{3n}\right).$$

Proof. By putting m=1, 2, 3 and 4 in equation (4.38) we get the above results respectively. □

4.4. Summary of infinite mixture of *Maxwell* distribution

The *Tau square* distribution is a *chi square* like distribution with similar properties as *chi square*. When $\mathrm{T}^2 = X/\nu$ and $\nu = 3n$ then *tau square* distribution reduces to chi square distribution with ν degrees of freedom. The distribution is derived from the MLE of *Maxwell* distribution. This distribution has been mixed with the *Maxwell* distribution.

In this chapter, we mainly focused on different characteristics of *tau square* mixture of *Maxwell* distribution. Among these properties MGF, CF, survival function and hazard function are specifically discussed. Beside this, the *m*-th moments of the distribution discussed as well. In the next chapter, we will continue discussing this new distribution in light of estimating its distributional parameters.

CHAPTER FIVE

PARAMETER ESTIMATION OF MIXTURE MAXWELL DISTRIBUTIONS

Parameters are called the characteristic of population. Since it is not feasible to measure an entire population and in practice the value of the parameters are unknown, hence we need to estimate these parameters. For this purpose, we need to take a random sample from the desired population. One of the objectives of statistical analyses is to obtain estimates of these population parameters. The estimates of the population parameters are called sample statistics as they are calculated for a random sample.

In this chapter we present different estimation techniques of population parameters for mixture of Maxwell distributions. More specifically, maximum likelihood estimation and method of moments are discussed.

5.1. MLE of the parameters of finite mixture Maxwell distribution

Suppose that a set of probability density function (PDF) is $f_1\left(r;\sigma_1\right)$, $f_2\left(r;\sigma_2\right)$, ..., $f_k\left(r;\sigma_k\right)$ with corresponding cumulative distribution functions (CDF) $F_1\left(r;\sigma_1\right), F_2\left(r;\sigma_2\right), ..., F_k\left(r;\sigma_k\right)$ respectively. Then the k component mixture distribution can be presented as

$$f\left(r;\sigma_i,\theta_i\right)=\sum_{i=1}^{k}\theta_i f_i\left(r;\sigma_i\right), \tag{5.1}$$

for $r>0$, $\sigma_i>0$, $0\le\theta_i\le1$, $\left(i=1,2,...,k\right)$, $\sum_{i=1}^{k}\theta_i=1$ and assume without loss of generality that $\sigma_1>\sigma_2>...>\sigma_k$.

Let $r_1,r_2,...,r_n$ be a random sample of n observations from a population with PDF $f\left(r;\sigma_i,\theta_i\right)$ given in equation (5.1). Then the data structure of the mixture distribution is given in the following **Table 5.1**.

Table 5.1 Data structure of mixture distribution

		Mixing Proportion (θ)					
		θ_1	θ_2	$\cdots$	θ_i	$\cdots$	θ_k
Sample Observation	r_1	f_{11}	f_{12}	$\cdots$	f_{1i}	$\cdots$	f_{1k}
	r_2	f_{21}	f_{22}	$\cdots$	f_{2i}	$\cdots$	f_{2k}
	r_3	f_{31}	f_{32}	$\cdots$	f_{3i}	$\cdots$	f_{3k}
	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	r_j	f_{j1}	f_{j2}	$\cdots$	f_{ji}	$\cdots$	f_{jk}
	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	r_n	f_{n1}	f_{n2}	$\cdots$	f_{ni}	$\cdots$	f_{nk}

The joint probability density function of these sample observations is

$$L\left(r_1,r_2,..,r_n\ ;\Theta_i\right)=\prod_{j=1}^{n}f_j\left(r;\sigma_i,\theta_i\right)=\prod_{j=1}^{n}f_j\left(r;\Theta_i\right). \tag{5.2}$$

where, $\Theta_i=\left(\sigma_i,\theta_i\right)$. Here the joint density function is also known as likelihood function and usually denoted by L. Hence, using equation (5.1), the likelihood function for k component mixture distribution is

$$L\left(\Theta_i\right)=L\left(r_1,r_2,..,r_n\ ;\Theta_i\right)=\prod_{j=1}^{n}\sum_{i=1}^{k}\theta_i f_{ji}\left(r;\sigma_i\right)$$

$$\text{So, } L\left(\Theta_i\right)=\prod_{j=1}^{n}\left\{\theta_1 f_{j1}\left(r;\sigma_1\right)+\theta_2 f_{ji}\left(r;\sigma_2\right)+...+\theta_k f_{jk}\left(r;\sigma_k\right)\right\}, \tag{5.3}$$

and the log likelihood function is

$$\log L\left(\Theta_i\right)=\log\prod_{j=1}^{n}f_j\left(r;\Theta_i\right)=\sum_{j=1}^{n}\log f_j\left(r;\Theta_i\right)=\sum_{j=1}^{n}\log f_j\left(r;\sigma_i,\theta_i\right). \tag{5.4}$$

To obtain MLE we solve following equations $\frac{\partial \log L}{\partial \sigma_i} = 0$ and $\frac{\partial \log L}{\partial \theta_i} = 0.$

In particular,

$$\frac{\partial}{\partial \sigma_i} \log L(\Theta_i) = \frac{\partial}{\partial \sigma_i} \sum_{j=1}^{n} \log f_j(r;\sigma_i,\theta_i)$$

$$= \sum_{j=1}^{n} \frac{\partial}{\partial \sigma_i} \log f_j(r;\sigma_i,\theta_i)$$

$$= \sum_{j=1}^{n} \frac{1}{f_j(r;\sigma_i,\theta_i)} \frac{\partial}{\partial \sigma_i} f_j(r;\sigma_i,\theta_i)$$

$$= \sum_{j=1}^{n} \frac{1}{f_j(r;\sigma_i,\theta_i)} \frac{\partial}{\partial \sigma_i} \left(\sum_{i=1}^{k} \theta_i f_{ji}(r;\sigma_i) \right)$$

$$= \sum_{j=1}^{n} \frac{1}{f_j(r;\sigma_i,\theta_i)} \frac{\partial}{\partial \sigma_i} \left(\theta_i f_{ji}(r;\sigma_i) \right).$$

so, $$\frac{\partial}{\partial \sigma_i} \log L(\Theta_i) = \theta_i \sum_{j=1}^{n} \frac{1}{f_j(r;\alpha_i,\theta_i)} \frac{\partial}{\partial \sigma_i} \left(f_{ji}(r;\sigma_i) \right). \tag{5.5}$$

Also,

$$\frac{\partial}{\partial \theta_i} \log L(\Theta_i) = \sum_{j=1}^{n} \frac{\partial}{\partial \theta_i} \log f_j(r;\sigma_i,\theta_i)$$

$$= \sum_{j=1}^{n} \frac{1}{f_j(r;\sigma_i,\theta_i)} \frac{\partial}{\partial \theta_i} f_j(r;\sigma_i,\theta_i)$$

$$= \sum_{j=1}^{n} \frac{1}{f_j(r;\sigma_i,\theta_i)} \frac{\partial}{\partial \theta_i} \left(\sum_{i=1}^{k} \theta_i f_{ji}(r;\sigma_i) \right)$$

$$= \sum_{j=1}^{n} \frac{f_{ji}(r;\sigma_i)}{f_j(r;\sigma_i,\theta_i)} \frac{\partial}{\partial \theta_i} (\theta_i).$$

So, $$\frac{\partial}{\partial \theta_i} \log L(\Theta_i) = \sum_{j=1}^{n} \frac{f_{ji}(r;\sigma_i)}{f_j(r;\sigma_i,\theta_i)}. \tag{5.6}$$

Hence, using equations (5.5) and (5.6) normal equations can be written as,

$$\theta_i \sum_{j=1}^{n} \frac{f'_{ji}(r;\alpha_i)}{f_j(r;\alpha_i,\theta_i)} = 0, \tag{5.7}$$

$$\sum_{j=1}^{n} \frac{f_{ji}(r;\sigma_i)}{f_j(r;\sigma_i,\theta_i)} = 0. \tag{5.8}$$

As analytical solution of these normal equation is cumbersome. Thus, we have to move alternative way to solve these equations. EM algorithm is one of the most prominent numerical method of MLE estimation technique. In the next subsection we describe the EM algorithm for mixture distribution.

5.1.1. The EM algorithm

In this subsection we discuss the EM algorithm in general first. Then we develop a method for finite mixture distribution. After that we implement the developed method for finite mixture of *Maxwell* distribution. In general an EM algorithm contains the following steps

1. Define missing and complete data.
2. Calculate the conditional expectation of the complete log-likelihood given the observed data using some initial estimates. This is called E (expectation) step.
3. Maximize the corresponding Q-function to obtain a new estimate. This is called M (maximization) step.
4. Iteratively replace the initial estimate with the new estimate in step 2 and repeat step 2 and 3 until a stopping criterion is reached.

5.1.2. The EM algorithm for finite mixture distribution

Let us define a vector of missing observation $z = (z_1, z_2, ..., z_n)$ such that each element of z vector is either one or zero depending on that the *j*-th observation comes from the *i*-th component or not. That is, the element of z is defined as

$$z_{ij} = \begin{cases} 1; & \text{if } r_j \text{ belongs to the } i \text{ th component} \\ 0; & \text{if } r_j \text{ does not belong to the } i \text{ th component} \end{cases}$$

for $j = 1, 2, ..., n$ and $i = 1, 2, ..., k$. The corresponding indicator random variable is denoted by Z and its mass function by $\tilde{g}(z, \Theta)$. By defining the complete-data vector as $u = (r, z)$

and denoting $h(u)=h(r,z,\Theta)$ as joint density function of the random variable U, the complete log-likelihood may be written as

$$\begin{aligned}
\log L_c(\Theta,u) &= \log\prod_{j=1}^{n} h(u_j;\Theta) = \sum_{j=1}^{n}\log h(r_j,z_j;\Theta) \\
&= \sum_{j=1}^{n}\log\left(f_j(r_j;\Theta\mid z_j).\tilde{g}(z_j;\Theta)\right) \\
&= \sum_{j=1}^{n}\log\left(\prod_{i=1}^{k} f(r_j;\sigma_i)^{z_{ij}}\prod_{i=1}^{k}\theta_i^{z_{ij}}\right) \\
&= \sum_{j=1}^{n}\log\left(\prod_{i=1}^{k}\theta_i^{z_{ij}} f(r_j;\sigma_i)^{z_{ij}}\right) \\
&= \sum_{j=1}^{n}\log\left(\prod_{i=1}^{k}\left(\theta_i f(r_j;\sigma_i)\right)^{z_{ij}}\right) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{k}\log\left(\theta_i f(r_j;\sigma_i)\right)^{z_{ij}}.
\end{aligned}$$

$$\text{So, } \log L_c(\Theta,u) = \sum_{i=1}^{k}\sum_{j=1}^{n} z_{ij}\log\left(\theta_i f(r_j;\sigma_i)\right). \tag{5.9}$$

The E-step

In this step, we calculate conditional expectation of the complete log-likelihood given the observe data r using $\Theta^{(t)}$. Let the conditional expectation be denoted by $Q\left(\Theta,\Theta^{(t)}\right)$ and defined as

$$\begin{aligned}
Q\left(\Theta,\Theta^{(t)}\right) &= E_{\Theta^{(t)}}\left[\log L_c(\Theta,U)\mid R=r\right], \\
&= E_{\Theta^{(t)}}\left[\sum_{i=1}^{k}\sum_{j=1}^{n} Z_{ij}\log\left(\theta_i f(r_j;\sigma_i)\right)\mid R=r\right].
\end{aligned}$$

$$\text{So, } Q\left(\Theta,\Theta^{(t)}\right) = \sum_{i=1}^{k}\sum_{j=1}^{n}\left\{\log\left(\theta_i f(r_j;\sigma_i)\right)\right\} E_{\Theta^{(t)}}\left[Z_{ij}\mid R_j=r_j\right]. \tag{5.10}$$

Now using the law of total probability in the Bayes theorem the conditional expectation part shown in equation (5.10) can be written as

$$\begin{aligned}
E_{\Theta^{(t)}}\left[Z_{ij}\mid R_j=r_j\right] &= 0\times P\left(Z_{ij}=0\mid R_j=r_j\right)+1\times P\left(Z_{ij}=1\mid R_j=r_j\right) \\
&= P\left(Z_{ij}=1\mid R_j=r_j\right) \\
&= \frac{P\left(R_j=r_j\mid Z_{ij}=1\right)P\left(Z_{ij}=1\right)}{\sum_{i=1}^{k}P\left(R_j=r_j\mid Z_{ij}=1\right)P\left(Z_{ij}=1\right)} \\
&= \frac{\theta_i^{(t)} f\left(r_j;\sigma_i^{(t)}\right)}{\sum_{i=1}^{k}\theta_i^{(t)} f\left(r_j;\sigma_i^{(t)}\right)}
\end{aligned}$$

$$= \frac{\theta_i^{(t)} f\left(r_j;\sigma_i^{(t)}\right)}{f\left(r_j;\Theta^{(t)}\right)}.$$

So, $E_{\Theta^{(t)}}\left[Z_{ij} \mid R_j = r_j\right] = e_{ij}^{(t)},$ (5.11)

where $e_{ij}^{(t)} = \frac{\theta_i^{(t)} f\left(r_j;\sigma_i^{(t)}\right)}{f\left(r_j;\Theta^{(t)}\right)}$; for $j = 1,2,...,n$ and $i = 1,2,...,k$. Hence, $e_{ij}^{(t)}$ represents the probability under $\Theta^{(t)}$ that the *j*-th observation belongs to the *i*-th component of the mixture given that $R_j = r_j$. Hence, (5.10) can be written as

$$\begin{aligned} Q\left(\Theta,\Theta^{(t)}\right) &= \sum_{i=1}^{k}\sum_{j=1}^{n}\left[\log\left(\theta_i f\left(r_j;\sigma_i\right)\right)\right]e_{ij}^{(t)} \\ &= \sum_{i=1}^{k}\sum_{j=1}^{n}\left[\log\left(\theta_i\right)+\log\left(f\left(r_j;\sigma_i\right)\right)\right]e_{ij}^{(t)}. \end{aligned}$$

So, $Q\left(\Theta,\Theta^{(t)}\right) = \sum_{i=1}^{k}\sum_{j=1}^{n} e_{ij}^{(t)} \log\theta_i + \sum_{i=1}^{k}\sum_{j=1}^{n} e_{ij}^{(t)} \log f\left(r_j;\sigma_i\right).$ (5.12)

The M-step

The M-step now requires the maximization of the *Q*-function with respect to Θ. Since θ_i appears only in the first term and σ_i only in the second term of the right hand side of (5.12), the maximization can be done separately. Starting with the maximization of the first term, it is necessary to solve

$$\frac{\partial}{\partial\theta_i}\left(\sum_{i=1}^{k}\sum_{j=1}^{n} e_{ij}^{(t)}\log\theta_i + \lambda\left[\sum_{i=1}^{k}\theta_i - 1\right]\right) = 0, \text{ and} \tag{5.13}$$

$$\frac{\partial}{\partial\lambda}\left(\sum_{i=1}^{k}\sum_{j=1}^{n} e_{ij}^{(t)}\log\theta_i + \lambda\left[\sum_{i=1}^{k}\theta_i - 1\right]\right) = 0, \tag{5.14}$$

where λ denotes a Lagrange multiplier, since the constraint $\sum_{i=1}^{k}\theta = 1$ needs to hold. This yields

$$\sum_{i=1}^{k}\frac{1}{\theta_i}\sum_{j=1}^{n} e_{ij}^{(t)} + \lambda k = 0. \tag{5.15}$$

Equation (5.15) can also be written, using equation (5.11), as

$$\lambda = -\frac{1}{k}\sum_{i=1}^{k}\frac{1}{\theta_i}\sum_{j=1}^{n}\frac{\theta_i f\left(r_j;\sigma_i\right)}{f\left(r_j;\Theta_i\right)}.$$

Since, $\sum_{i=1}^{k}\theta_i = 1$, the above equation can be simplifies to $\lambda = -n$. Substituting this in to (5.15) we get $\theta_i = \frac{1}{n}\sum_{j=1}^{n} e_{ij}^{(t)}$. Hence an iterative solution for θ_i is

$$\theta_i^{(t+1)} = \frac{1}{n}\sum_{j=1}^{n} e_{ij}^{(t)}. \tag{5.16}$$

Again, equation (5.14) gives $\sum_{i=1}^{k}\theta_i = 1$.

The maximization of (5.12) with respect to σ_i depends on the density function $f\left(r_j;\sigma_i\right)$.

Now we have to take 1st derivative and equate to zero to get iterative solution of σ_i.

5.1.3. The EM algorithm for finite *Maxwell* mixture distribution

In this subsection we implement the above two steps such as E-step and M- step for finite mixture *Maxwell* distribution. In this regard we have to recall some results we already derived in Chapter three. Particularly, the PDF of *k* component mixture *Maxwell* distribution is required which is given in equation (3.13).

The E-step

Applying equation (5.12) for the *Maxwell* mixture distribution, we have the conditional expectation as follows:

$$Q\left(\Theta,\Theta^{(t)}\right) = \sum_{i=1}^{k}\sum_{j=1}^{n} e_{ij}^{(t)}\log\theta_i + \sum_{i=1}^{k}\sum_{j=1}^{n} e_{ij}^{(t)}\left(\log\left(\sqrt{\frac{2}{\pi}}r_j^2\right) - 3\log\left(\sigma_i\right) - \frac{r_j^2}{2\sigma_i^2}\right).$$

The M-step

Now we have to take 1st derivative of the equation obtained in E-step and equate to zero to get iterative solution of σ_i.

$$\frac{\partial}{\partial\sigma_i}\left(\sum_{i=1}^{k}\sum_{j=1}^{n}e_{ij}^{(t)}\left(\log\left(\sqrt{\frac{2}{\pi}}r_j^2\right)-3\log(\sigma_i)-\frac{r_j^2}{2\sigma_i^2}\right)\right)=0,$$

$$\text{or, } -\sum_{i=1}^{k}\sum_{j=1}^{n}e_{ij}^{(t)}\frac{3}{\sigma_i}-\sum_{i=1}^{k}\sum_{j=1}^{n}e_{ij}^{(t)}\left(\frac{2r_j^2}{2\sigma_i^3}\right)=0,$$

$$\text{or, } \sum_{i=1}^{k}\sum_{j=1}^{n}e_{ij}^{(t)}\left(\frac{r_j^2}{\sigma_i^3}\right)=\sum_{i=1}^{k}\sum_{j=1}^{n}e_{ij}^{(t)}\frac{3}{\sigma_i}.$$

Therefore, the i-th component's relation can be written as

$$\sum_{j=1}^{n}e_{ij}^{(t)}\left(\frac{r_j^2}{\sigma_i^3}\right)=\sum_{j=1}^{n}e_{ij}^{(t)}\frac{3}{\sigma_i},$$

$$\text{or, } \sigma_i^2=\sum_{j=1}^{n}r_j^2e_{ij}^{(t)}\Big/\sum_{j=1}^{n}3e_{ij}^{(t)},$$

Therefore,
$$\sigma_i^{(t+1)}=\left(\sum_{j=1}^{n}r_j^2e_{ij}^{(t)}\Big/\sum_{j=1}^{n}3e_{ij}^{(t)}\right)^{\frac{1}{2}}, \tag{5.17}$$

$$\text{where } e_{ij}^{(t)}=\frac{\theta_i^{(t)}f\left(r_j;\sigma_i^{(t)}\right)}{\sum_{i=1}^{k}\theta_i^{(t)}f\left(r_j;\sigma_i^{(t)}\right)}.$$

We have to iterate E and M steps until convergence. That is if $\left|\frac{\Theta_i^{(t+1)}-\Theta_i^{(t)}}{\Theta_i^{(t+1)}}\right|<\varepsilon$, where the convergence tolerance ε can be chosen as $\varepsilon=1e^{-08}$ and $\Theta_i=\{\theta_i,\sigma_i\}$, for $\forall i$.

5.2. Method of moment estimation of parameter of infinite mixture of Maxwell distribution

Let, $R_1,R_2,...,R_n$ be a random sample of size n follows the *tau square* mixture of *Maxwell* distribution given in Chapter four. The first sample raw moment is defined as

$$m_1'=\frac{1}{n}\sum_{i=1}^{n}r_i=\bar{r}. \tag{5.18}$$

We already derived in Chapter four that the first raw moment of the *tau square* mixture of *Maxwell* distribution is

$$\mu_1'=\frac{4\sigma}{\sqrt{3n}\mathrm{B}(3n/2,1/2)}. \tag{5.19}$$

Now equating the equations (5.18) and (5.19) we get the estimate of the parameter σ as

$$\hat{\sigma} = \frac{\overline{r}}{4}\sqrt{3n}\,\mathrm{B}\left(3n/2, 1/2\right). \tag{5.20}$$

5.3. Summary of the parameter estimation of *Maxwell* mixture distributions

Estimation is a crucial issue in any statistical analysis. Analytical form of the estimator is always preferable. But sometimes, it is really difficult to have this analytical form. Since, estimation is still essential we have to go for alternative method of estimation called numerical method of estimation. One of the most important numerical method of estimation is the EM algorithm. In this chapter we demonstrated the EM algorithm in terms of *k* component *Maxwell* mixture distribution. Another well known method of estimation is method of moment. We also implemented this method for *tau square* mixture of *Maxwell* distribution.

As it is always anticipated to have application for any method proposed, by considering this issue different possible application of *k* component *Maxwell* mixture distribution and *tau square* mixture of *Maxwell* distribution are given in the next chapter. The application of regular *Maxwell* distribution has also been discussed in the field named statistical process control.

CHAPTER SIX

APPLICATION OF THE MAXWELL MIXTURE DISTRIBUTION

In the chapter we discuss possible application of the *Maxwell* mixture of distributions in terms of finite and infinite cases. The results which have been derived in the previous chapter also have been used in this regard. Some real-life phenomena are presented along with simulation study as well.

6.1. Application of *Maxwell* distribution in process monitoring

The *Maxwell* distribution has been first introduced in the field of statistical process control by [52]. Statistical process control (SPC) is a method of quality control which is widely used in industry to monitor the process by using statistical tools. The quality control concept in manufacturing was first conceived by Walter A. Shewhart in 1920. While working in the Bell Telephone Laboratories, he conducted research on methods to improve quality and came up with the SPC [54]. By considering the hypothesis whether the process is in statistical control or not, SPC guides the decision about the quality of the process. For this purpose, different tools are used. Its seven major tools are Histogram and stem-and-leaf plot, Check sheet, Pareto chart, Cause-and-effect diagram, Defect concentration diagram, Scatter diagram and Control chart [55]. Among all of the above tools, the Shewhart control chart is the most technically sophisticated.

In the literature, the Shewhart control charts are widely used to monitor variation/shift in process quality characteristic of location as well as dispersion. [56] introduced *improved R chart* (IRC) and *improved S chart* (ISC) for monitoring the process variance. The typical Shewhart control charts are constructed based on the assumption that observed data to monitor the quality characteristics of the process is from normal or near-normal distribution [57].

But in real life there are many situations that do not meet the normality assumption rather that follow skewed distribution. A number of researchers studied quality monitoring in this context. [58], observed the quality characteristic of the compressive strength (kgf/cm2) of concrete under log-normal distribution. [59], studied the control chart for skewed distributions including Weibull, Gamma, and log-normal distributions. [60], studied control chart for location parameter of the lognormal process. In lifetime data applications, [61] studied the Weibull distribution under type II censoring for monitoring shape parameter.

According to our knowledge, as one of the non-normal skewed distributions, the *Maxwell* distribution which has immense application in statistical mechanics, chemistry as well as in lifetime modelling [62], [63], [64] has not been studied yet in the field of process control. The *Maxwell* distribution has only one parameter which is the scale parameter given in equation (1.1) in Chapter one.

6.1.1. Derivation of the distribution of V (the estimate of scale parameter)

In this section, we develop the theory for the construction of the SPC chart for the *Maxwell* parameter.

Theorem 6.1 Given the PDF in (1.1), the transformation $T = R^2 / (2\sigma^2)$ has the PDF of a $gamma(3/2,1)$ distribution.

Proof. In the *Maxwell* PDF given in equation (1.1) in Chapter one, let $t = r^2/(2\sigma^2)$, which implies that $r = \sigma\sqrt{2t}$. Then the transformation Jacobian, $J(r \to t) = dr/dt = \sigma/\sqrt{2t}$. Hence the distribution of *t* can be found by the equation $f_T(t) = f_R(t)|J| = (2/\sqrt{\pi})t^{(1/2)}e^{-t}$ and finally

$$f_T(t) = \frac{1}{\Gamma(3/2)} t^{(3/2)-1} e^{-t}, \tag{6.1}$$

which is the PDF of gamma distribution with parameters as given by the theorem. □

Symbolically, $T \sim gamma(3/2,1)$, or, $R^2/2\sigma^2 \sim gamma(3/2,1)$.

Theorem 6.2 Given a random sample of size *n* from the PDF in equation (1.1) and the transformation $T = R^2/(2\sigma^2)$ and the random variable $V = \hat{\sigma}^2$, the random variable $U = 3nV/(2\sigma^2)$ has the PDF of a $gamma(3n/2,1)$ distribution.

Proof. From the additive rule of gamma distribution [65], if $X_1 \sim gamma(\alpha_1, \beta)$ and $X_2 \sim gamma(\alpha_2, \beta)$ then $X_1 + X_2 \sim gamma(\alpha_1 + \alpha_2, \beta)$. In general, if $X_i \sim gamma(\alpha_i, \beta)$ then $\sum_{i=1}^n X_i \sim gamma\left(\sum_{i=1}^n \alpha_i, \beta\right)$. In our case, since, $T_i = R_i^2/(2\sigma^2) \sim gamma(3/2,1)$, $\sum_{i=1}^n T_i \sim gamma(3n/2,1)$. Now recall equation of MLE and replace $\hat{\sigma}^2$ with *V*. Then $V = (3n)^{-1}\sum_{i=1}^n r_i^2 \Rightarrow 3nV = \sum_{i=1}^n r_i^2$. Finally,

$$3nV/(2\sigma^2) = \sum_{i=1}^n r_i^2/(2\sigma^2) = \sum_{i=1}^n t_i. \tag{6.2}$$

Now, let the left hand side of equation (6.2) be denoted by *U*. Hence due to the additive rule of gamma variates with common β parameter, $U = 3nV/(2\sigma^2)$ is a pivotal quantity which follows $gamma(3n/2,1)$ distribution. □

Recall that the mean of the gamma variate, U, is

$$E(U)=E\left[3nV/\left(2\sigma^2\right)\right]=3n/2 \Rightarrow E\left(V/\sigma^2\right)=1.$$

Therefore, $$E(V)=\sigma^2. \tag{6.3}$$

which says that V is an unbiased estimator of σ^2. Recall also that the variance of the *gamma* variate is $Var(U)=Var\left[3nV/\left(2\sigma^2\right)\right]=3n/2 \Rightarrow \left[3n/\left(2\sigma^4\right)\right]Var(V)=1$.

Therefore, $$Var(V)=2\sigma^4/(3n). \tag{6.4}$$

It is also known from the literature that the CDF of $X \sim gamma(\alpha,\beta)$ is $F(x)=\left[1/\Gamma(\alpha)\right]\gamma(\alpha,x/\beta)$ [65]. Hence the CDF of the pivotal quantity, $U=3nV/\left(2\sigma^2\right) \sim gamma(3n/2,1)$ is $F(u)=\left[1/\Gamma(3n/2)\right]\gamma(3n/2,u)$ where $\gamma(.,.)$ is the incomplete gamma function defined in equation (1.3). Hence, the $\alpha\, th$ quantile can be found as $F(u)=\alpha \Rightarrow U=F^{-1}(\alpha) \Rightarrow 3nV/\left(2\sigma^2\right)=F^{-1}(\alpha)$.

Therefore, $$V_\alpha=\left[2\sigma^2/(3n)\right]F^{-1}(\alpha). \tag{6.5}$$

6.1.2. Process Monitoring

To monitor the scale parameter σ^2 we use the estimate V, where the terms *LPL*, *UPL* are used as *lower probability limit* and *upper probability limit* respectively in case of probability limits and *LCL*, *CL* and *UCL* are termed as *lower control limit, central line* and *upper control limit* in case of *L*-sigma limits. Now we have to test whether any shift occurs in the process or not. To do so we need to set the following hypothesis.

$$H_0:\sigma^2=\sigma_0^2;\ or,\ \delta=1,\ i.e.\ \text{No shift occurs in the process.}$$
$$H_1:\sigma^2=\sigma_1^2=\delta\sigma_0^2;\ or,\ \delta\neq 1, i.e.\ \text{Shift occurs in the process.}$$

Here, δ represents shift in the process. Recall equation (6.5) to construct the probability limits for V which can be shown as

$$LPL: V_{(\alpha/2)} = \left[2\sigma^2/(3n)\right]F^{-1}(\alpha/2), \text{ and}$$
$$UPL: V_{1-(\alpha/2)} = \left[2\sigma^2/(3n)\right]F^{-1}(1-\alpha/2).$$

These can also be presented as

$$LPL: V_{(\alpha/2)} = L_1\sigma^2, \text{ and}$$
$$UPL: V_{1-(\alpha/2)} = L_2\sigma^2,$$

where $L_1 = \left[2/(3n)\right]F^{-1}(\alpha/2)$ and $L_2 = \left[2/(3n)\right]F^{-1}(1-\alpha/2)$. These coefficients are simply quantiles from the gamma distribution multiplied by some constants. For different sample size *n* and different false alarm rate α these coefficients vary. **Table 6.1** shows an illustration of the variations in quantiles.

Table 6.1: Gamma quantiles for different *n* and *α*

Sample size *(n)*	**False alarm rate α**					
	0.005		0.0027		0.002	
	L_1	L_2	L_1	L_2	L_1	L_2
1	0.0150	4.7734	0.0099	5.2094	0.0081	5.4221
2	0.0878	3.3749	0.0706	3.6228	0.0635	3.7430
3	0.1611	2.8292	0.1380	3.0101	0.1280	3.0975
4	0.2218	2.5265	0.1959	2.6722	0.1845	2.7425
5	0.2713	2.3300	0.2442	2.4536	0.2322	2.5132

In practical cases two situations arise in process monitoring of the *Maxwell* scale parameter: i) σ^2 known and ii) σ^2 unknown. In case of known σ^2**,** the limits can be written as

$$LPL = L_1\sigma_0^2; \qquad CL = \sigma_0^2 \quad \text{and} \quad UPL = L_2\sigma_0^2. \tag{6.6}$$

But in case of unknown σ^2, we have to estimate *V* as and use this estimate to monitor the next process

$$LPL = L_1\overline{V}; \qquad CL = \overline{V} \quad \text{and} \quad UPL = L_2\overline{V}. \tag{6.7}$$

where, $\overline{V}$ is the arithmetic mean of estimated V obtained in each of the sample over time. To check the performance of the chart, average run length (ARL) and power of the test related to the chart are two helpful tools to be examined. ARL is defined as,

$$ARL = 1/(1-\beta), \tag{6.8}$$

where $\beta = (1-\text{power})$.

The conventional definition of power of a test is the probability of rejecting null hypothesis (H_0) when alternative hypothesis (H_1) is true. That is, $\text{Power} = \Pr(\text{reject } H_0 \mid H_1)$. In our case, the null hypothesis is rejected when the plotting statistic (V) is placed either below the lower probability limit or above the upper probability limit. Hence,

$$\text{Power} = \Pr(V_\alpha < LPL_0 \mid H_1) + \Pr(V_\alpha > UPL_0 \mid H_1)$$

which implies that,

$$\text{Power} = \Pr\left(V_\alpha < \left(2\sigma_0^2/3n\right)F^{-1}(\alpha/2) \middle| \delta \neq 1\right) + \Pr\left(V_\alpha > \left(2\sigma_0^2/3n\right)F^{-1}(1-\alpha/2) \mid \delta \neq 1\right),$$

where $F^{-1}(\alpha) = 3nV/\left(2\sigma^2\right)$. Finally, power of the chart is obtained as follows:

$$\begin{aligned} \text{power} = 1 &+ \left[\Gamma(3n/2)\right]^{-1} \gamma\left[3n/2, \delta^{-1}F^{-1}(\alpha/2)\right] \\ &- \left[\Gamma(3n/2)\right]^{-1} \gamma\left[3n/2, \delta^{-1}F^{-1}(1-\alpha/2)\right]. \end{aligned} \tag{6.9}$$

In equation (6.9) if $\delta = 1$, that is no shift in the process, then the power is equivalent to false alarm rate α. **Figure 6.1** below shows the power for different process shifts. Here power is increasing when sample size increase. The chart performs better for larger shifts as well. That is, the larger the shifts the higher the power.

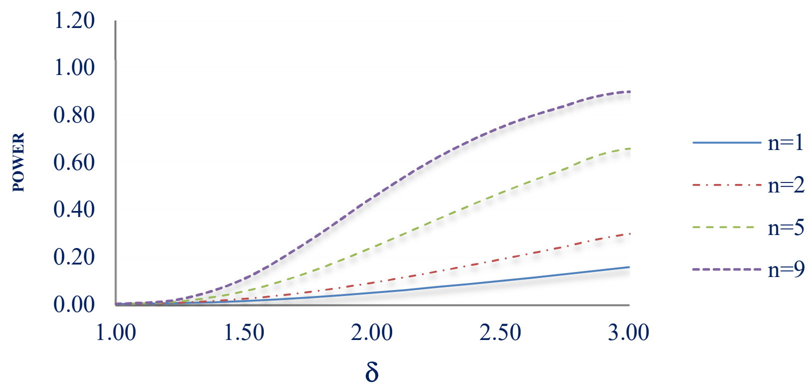


Figure 6.1: Power curves of control charts for different n at $\alpha = 0.0027$

Using the power and *ARL* relationship shown in equations (6.8) and (6.9),**hen sample size** is 2 and so on.

Table 6.2 displays *ARL* for corresponding power and shifts. It is clear from the table that when the process is in control, that is $\delta = 1$, then we have to wait up to 370 samples to get a false alarm which is fixed for any sample size. Similarly, a shift of amount 1.25 will be detected after 147 samples when sample size is 1, after 115 samples when sample size is 2 and so on.

Table 6.2: *ARL* of control charts for different *n* at α = 0.0027

Shift (δ)	**Sample size (*n*)**			
	1	**2**	**5**	**9**
1.00	370	370	370	370
1.25	147	115	69	43
1.50	62	40	18	9
1.75	32	19	7	4
2.00	20	11	4	2
2.25	14	7	3	2
2.50	10	5	2	1
2.75	8	4	2	1
3.00	6	3	2	1
6.00	2	1	1	1

Based on the mean and variance of V given in equations (6.3) and (6.4), L-sigma limits of V are given as follows:

$$LCL: E(V) - L \times SD(V) = \left[1 - L\sqrt{2/(3n)}\right]\sigma^2 = W_1\sigma^2, \tag{6.10}$$

$$CL: E(V) = \sigma^2, \text{ and} \tag{6.11}$$

$$UCL: E(V) + L \times SD(V) = \left[1 + L\sqrt{2/(3n)}\right]\sigma^2 = W_2\sigma^2, \tag{6.12}$$

where, $W_1 = \left[1 - L\sqrt{2/(3n)}\right]$ and $W_2 = \left[1 + L\sqrt{2/(3n)}\right]$. The factor L is obtained using the gamma quantile in such a way that the desired false alarm rate (α) has been attained.

In this case two situations again arise: i) σ^2 known and ii) σ^2 unknown. In case of known σ^2**,** the limits can be written as

$$LCL = W_1\sigma_0^2; \qquad CL = \sigma_0^2 \quad \text{and} \quad UCL = W_2\sigma_0^2.$$

But in case of unknown σ^2**,** we have to estimate V and use it as follows:

$$LCL = W_1\overline{V}; \qquad CL = \overline{V} \quad \text{and} \quad UCL = W_2\overline{V}.$$

Table 6.3 below represents L coefficients which is used to calculate the factors W_1 and W_2. The L coefficients are chosen from the quantile relationship given in equation (6.5) such that we achieve the fixed false alarm rate.

Table 6.3: L coefficients

Sample size (*n*)	**False Alarm Rate (α)**		
	0.005	**0.0027**	**0.002**
	L	***L***	***L***
2	0.3390	0.2700	0.2200
3	0.8690	0.7400	0.6541
4	1.5400	1.3490	1.2350
5	2.3000	2.0590	1.9845
6	3.1300	2.8530	2.7925
7	4.0210	3.6810	3.4549
8	4.9400	4.5600	4.4985
9	5.8990	5.4900	5.2460
10	6.8900	6.4432	6.3214

In addition, **Table 6.4** displays the corresponding W_1 and W_2 factors. One can use these factors directly to avoid the complexity of calculation involving the L factors. These W_1 and W_2 factors would be very helpful to construct control chart for *Maxwell* parameter easily.

Table 6.4: Factor for constructing Control chart for *Maxwell* parameter

Sample size (n)	False Alarm Rate (α)					
	0.005		0.0027		0.002	
	W_1	W_2	W_1	W_2	W_1	W_2
2	0.8043	1.1957	0.8441	1.1559	0.8730	1.1270
3	0.5903	1.4097	0.6512	1.3488	0.6917	1.3083
4	0.3713	1.6287	0.4493	1.5507	0.4958	1.5042
5	0.1602	1.8398	0.2482	1.7518	0.2754	1.7246
6	0.0000	2.0433	0.0490	1.9510	0.0692	1.9308
7	0.0000	2.2409	0.0000	2.1360	0.0000	2.0662
8	0.0000	2.4261	0.0000	2.3164	0.0000	2.2986
9	0.0000	2.6055	0.0000	2.4942	0.0000	2.4278
10	0.0000	2.7790	0.0000	2.6636	0.0000	2.6322

6.1.3. Simulation Study

To construct control chart for monitoring the *Maxwell* distribution parameter, we use the results from the preceding sections. We also shall call this chart as the V chart. In this section, we will discuss the construction of the chart with simulated data. Steps for generating data from *Maxwell* distribution and plotting the control chart are given below:

Step 1: Choose a number for a random sample of size n.

Step 2: Generate gamma random variable T of size n with parameter 3/2 and $1/(2\sigma_0^2)$.

Step 3: Calculate *Maxwell* random variable R of size n by taking square root on T.

Step 4: Obtain the plotting statistic V.

Step 5: Repeat step 1 to step 4 until the desired number of sample batches are attained.

Step 6: Construct the control limits as described in the previous section.

Step 7: Plot all V (at different sample batch number) against control limits.

To achieve this, we simulated data from the *Maxwell* distribution using statistical software *R* 3.2.2. For this purpose we use $\sigma_0 = 1777.86$. The obtained data were first cross checked using the Kolmogorov-Smirnov test [66] with the hypothesis of whether the data came from a *Maxwell* distribution or not. We fail to reject the hypothesis that, "the data came from *Maxwell* distribution", thus we support the hypothesis that the data follow the *Maxwell* distribution. Then we took 25 samples each of size 4 from the simulated data.

Now from the **Table 6.1** for fixed false alarm rate α=0.0027 and n=4, $L_1 = 0.1959$ and L_2 =2.6722, and known $\sigma_0 = 1777.86$, consequently, the probability limits are

$$LPL = 619095; \qquad CL = 3160782 \qquad \text{and} \qquad UPL = 8446328.$$

A graphical presentation of a V control chart based on these limits is given in the **Figure 6.2** below.

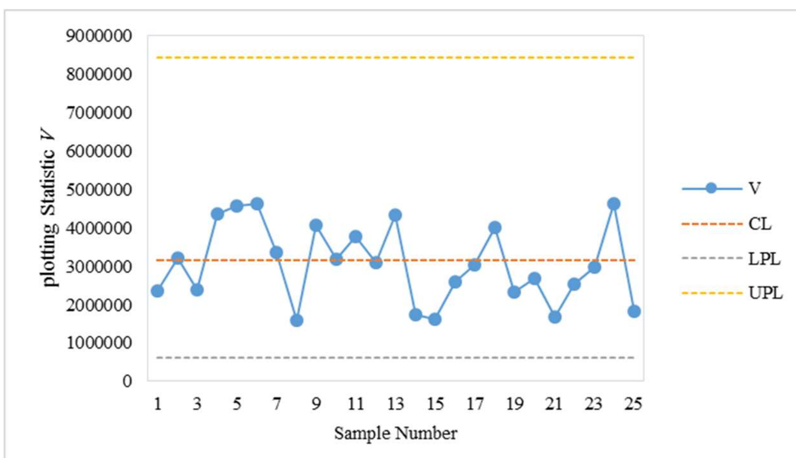


Figure 6.2: *V*-Chart for *Maxwell* Parameter using probability limits (in control situation)

Alternatively, for the same false alarm rate and sample size and know parameter $\sigma_0 = 1777.86$, the L sigma limits are as follows

$$LCL = 1420139; \qquad CL = 3160782 \qquad \text{and} \qquad UCL = 4901425.$$

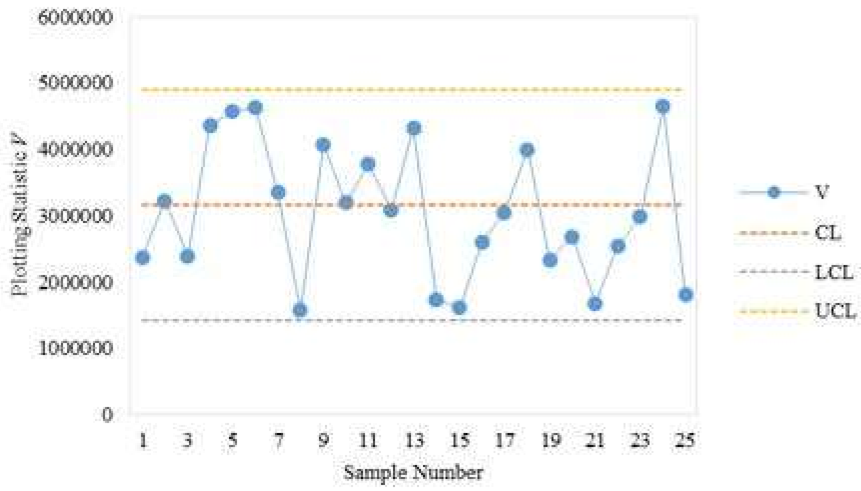


Figure 6.3: *V*-Chart for *Maxwell* Parameter using *L*-sigma limits (in control situation)

Figure 6.3 depicts the corresponding control chart for *V* based on these *L*-sigma limits. Both **Figure 6.2** and

Figure 6.3 show that the process is in control which support the theory. But in the current process *L*-sigma limits show less wide control limits as compared to probability limits.

In order to highlight the detection ability of our proposed chart for an out-of-control situation, we consider a shift in the process scale parameter. We assume that the process scale parameter σ^2 has been shifted to a new level such that $\delta = 2.25$ (after the 16th sample). For α=0.0027 and n=4, $L_1 = 0.1959$ and L_2=2.6722, and known $\sigma_0 = 1777.86$, the control charts are constructed for out-of-control scenario using both probability and *L* sigma limits. These resulting charts are presented in **Figure 6.4** and **Figure 6.5**.It is evident from these charts that there is an upward shift in the process, exhibiting a pattern (as shown in **Figure 6.4** and **Figure 6.5**), indicating that the process is in an out-of-control state.

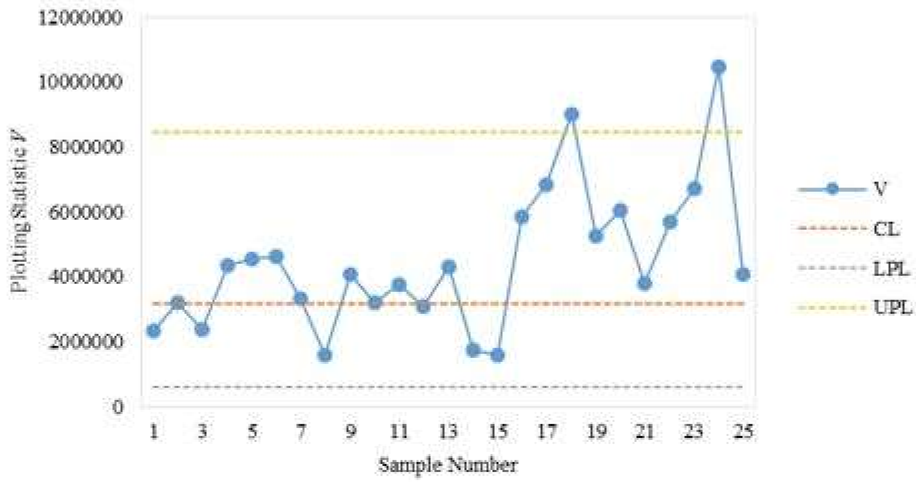


Figure 6.4: *V*-Chart for *Maxwell* Parameter using probability limits (in out of control situation)

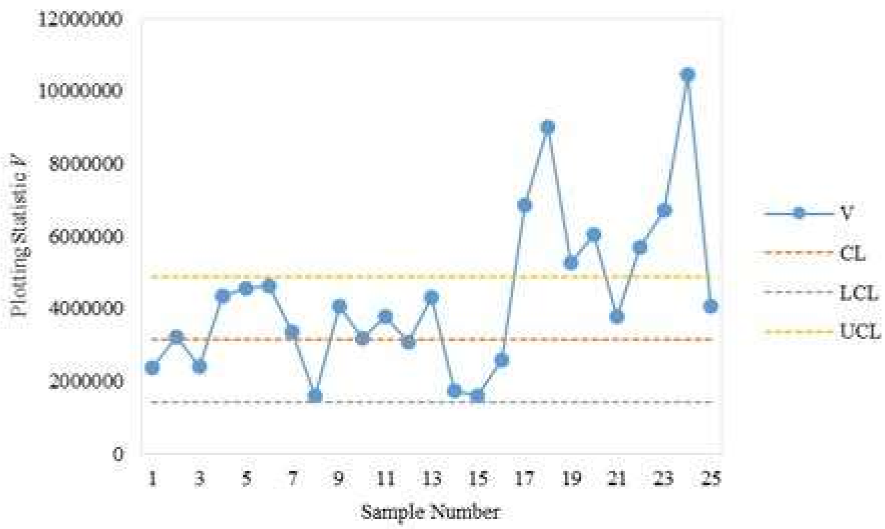


Figure 6.5: *V*-Chart for *Maxwell* Parameter using *L*-sigma limits (in out of control situation)

6.1.4. Real Life Example

For any data set, before starting analysis, the first job is to check normality assumption. Whether the data follows normal distribution or not. If it violates the normality assumption, then the next step is to check which non-normal distribution does it follow. For non-normal situation we may consider *Maxwell* distribution as well. For this purpose different goodness of fit test are available in the literature, for example, Kolmogorov-Smirnov test [66].

To illustrate the applicability of these control charts with real data, we use the data in [67]. The Kolmogorov-Smirnov test for this data produces a p-value of 0.4775 for the null hypothesis that the data came from a *Maxwell* distribution. Thus, we do not reject the hypothesis and conclude that the *Maxwell* distribution fit the data fairly well. We used this data to construct control chart for *Maxwell* distribution parameter. The data consist of failure of vertical boring machine (VBM) with 32 observations. We considered the data in the form of subgroups, each of size 4, that results into 8 subgroups. Then we use the subgroups further in the construction of the control charts. The failure times in hours are as follows:

Table 6.5: Failure of vertical boring machine (in hours)

Sample Number	**Observation**			
	1st	**2nd**	**3rd**	**4th**
1	2802	2937	2136	4359
2	4020	1781	2816	2655
3	3886	2296	3158	3695
4	4155	3811	2380	376
5	2172	3705	2848	4339
6	2076	2672	3632	1976
7	1700	1596	1701	3575
8	3802	4351	4291	808

Now we construct the control chart for V using probability limits and L-sigma limits from a *Maxwell* distribution. For our given data, the estimated value of the MLE is $\overline{V} = 3160782$. From the **Table 6.1** for fixed false alarm rate α=0.0027 and n=4, we have, $L_1 = 0.1959$ and L_2=2.6722. Hence, the probability limits are

$$LPL = 619197; \qquad CL = 3160782 \qquad \text{and} \qquad UPL = 8446242.$$

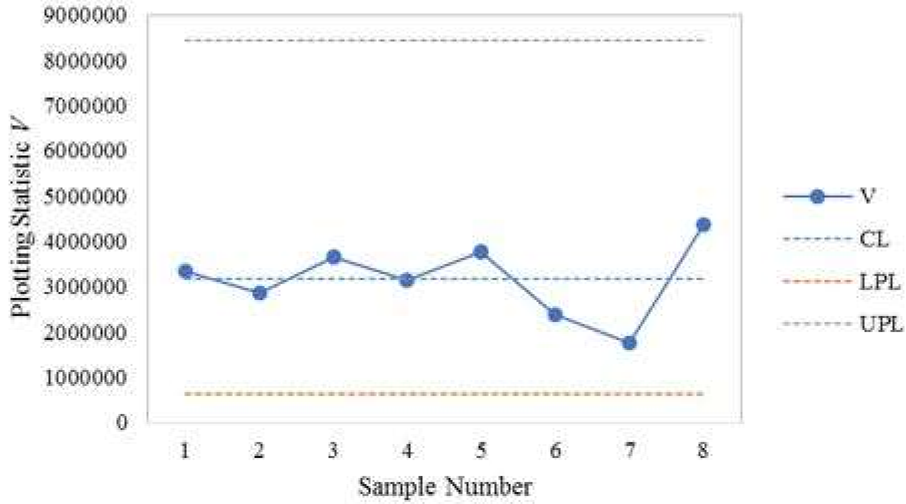


Figure 6.6: *V*-Chart for *Maxwell* Parameter using probability limits

Here, the *Maxwell* scale statistic is within the probability limits with no aberrant patterns as data points are hovering around the center line.

Again, from the **Table 6.4** for fixed false alarm rate α=0.0027 and n=4, W_1 = 0.4493 and W_2 =1.5507, the control limits are

$$LCL = 1420054; \qquad CL = 3160782 \qquad \text{and} \qquad UCL = 4901510.$$

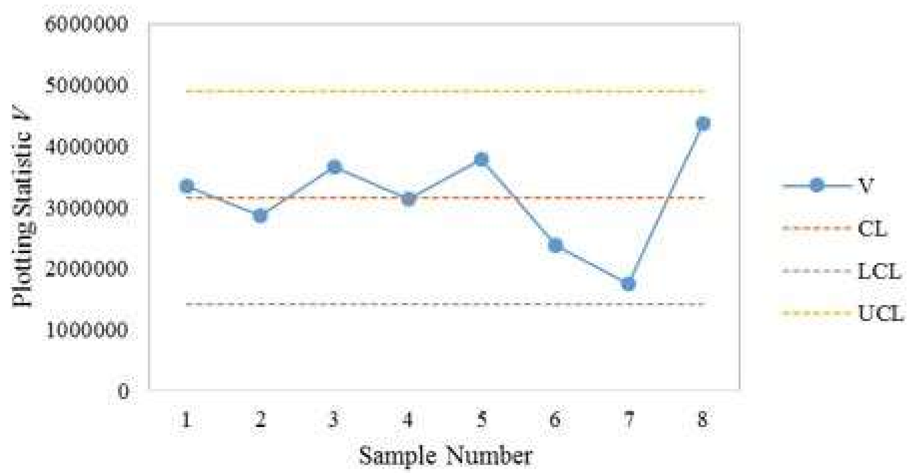


Figure 6.7: *V*-Chart for *Maxwell* Parameter using *L*-sigma limits

The data are all within the control limits signifying that the *Maxwell* scale parameter is within the expected ranges.

6.2. Application of finite mixture of *Maxwell* distribution in process monitoring

In statistical process monitoring various type of control charts are available. These chart can be classified into mainly two types such as variable control charts and attribute control charts. The variable control chart is used to monitor a process based on the assumption that the plotting statistics follows some continuous distributions whereas the attribute control chart is based on the assumption that the plotting statistic's distribution is discrete. To model the number of non-conformities or defective items or fraction of non-conformities, the very popular and well known distributions are *Binomial, Negative Binomial and Poisson* distributions.

The conventional method of monitoring a process, which follows above mentioned distributions, using control chart is Shewhart *c*-chart, *u*-chart, *p*-chart or *np*-chart. The *c*-chart and *u*-chart monitor the occurrence rate of non-conformities and *p*-chart and *np*-chart monitors the proportion of non-conformities in a process [55]. But, there is another way of monitoring non-conforming items in a process and is based on monitoring the time between events (TBE) [68]. Here the meaning of the terminologies "time" and "event" may depend on the specialty of the process. The occurrence of defective items in the manufacturing industry, failure to work of a machine in reliability engineering, death or occurrence of a particular disease in survival analysis etc. may refer to as "event". Whereas the term "time" may refer the attribute or variable that is observed between consecutive events of interest.

Moreover, in high production manufacturing process *c, u, p* and *np* chart don't work well. To overcome this problem [69], [70] proposed Cumulative Count Control (CCC), Cumulative Quantity Control (CQC) and Cumulative Probability Control (CPC) charts. In these charts the plotting statistic is cumulative count, cumulative quantity and cumulative probability until one or more defective items are inspected. Inspired by this and on some other research, [71] introduced MWQ chart for mixture *Weibull* distributed process to consider two types of defects when population of defective items consists of two sub population. For example, suppose that a VBM production company has two machines to produce VBMs. If a consumer finds a VBM which is defective, it may be from machine one or from the second machine. Therefore, total number of defective items from two different production machines may produce the population.

Here we proposed a new control chart based on these literatures. We named these as the MMCQ (Mixture *Maxwell* Cumulative Quantity) control chart for monitoring cumulative quantity between item non-conformities. This chart is mainly applicable in monitoring *Maxwell* distributed process, for example, to monitoring life time of vertical boring machine, based on mixture distribution. We also introduced MCQ chart based on regular *Maxwell* distribution and compared results with MMCQ control chart.

6.2.1. MCQ and MMCQ control charts

Let us assume that *R*, a continuous random variable referring to cumulative quantity of product between two defective items, follows *Maxwell* distribution with scale parameter σ. The CDF of the distribution is given by equation (1.2) in Chapter one. as

$$F(r) = \frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2}, \frac{r^2}{2\sigma^2}\right); \; r > 0. \tag{6.13}$$

Hence, the α *th* quantile of *R* can be found as

$$F(r) = \alpha \Rightarrow R_\alpha = F^{-1}_{Maxwell}(\alpha). \tag{6.14}$$

As mentioned in Theorem 6.1 the transformation $T = R^2/(2\sigma^2)$ follows gamma distribution with shape parameter 3/2 and scale parameter 1. Using this relationship the α th quantile of R can be written as

$$R_\alpha = \sigma\sqrt{2F_{gamma}^{-1}(\alpha)}. \tag{6.15}$$

Now, the two sided MCQ control chart can be constructed by equating (6.13) equal to $\alpha/2$ for *LPL*, equal to $1-\alpha/2$ for *UPL* and equal to $1/2$ for *CL* [69]. Therefore, the expression for limits can be shown as

$$LPL:\ R_L = \sigma\omega_L,\quad CL:\ R_C = \sigma\omega_C, \text{ and } UPL:\ R_U = \sigma\omega_U, \tag{6.16}$$

where, $\omega_L = \sqrt{2F_{gamma}^{-1}\left(\frac{\alpha}{2}\right)}$, $\omega_C = \sqrt{2F_{gamma}^{-1}\left(\frac{1}{2}\right)}$ and $\omega_U = \sqrt{2F_{gamma}^{-1}\left(1-\frac{\alpha}{2}\right)}$ and α is probability of getting signal or false alarm rate which is pre-specified. The one sided limits may also be presented in a similar fashion.

Corollary 6.1 The asymptotic representation of these control limits can be defined as follows: $\sigma \to 0$, $R_L = R_C = R_U \to 0$ and when $\sigma \to \infty$, $R_L = R_C = R_U \to \infty$.

The coefficients given in equation (6.16) are mainly from gamma distribution, with parameter 3/2 and 1, multiplied by some constant. The value of these coefficients depends on different values of α. The following table, may be from different countries and compiled in another country.

Table 6.6, shows different values of ω_L, ω_C and ω_U coefficients for various α.

Let us consider a practical situation where the final product is a combination of different ingredients from different plan or different company. For example, a VBM consists of different components such the electric motor, bearing, boring bar with cutter etc. These components may be from different countries and compiled in another country.

Table 6.6: Gamma quantiles for different False alarm rate (α)

α	ω_L	ω_C	ω_U
0.0100	0.2678	1.5382	3.5830
0.0067	0.2336	1.5382	3.7022
0.0050	0.2120	1.5382	3.7842
0.0040	0.1967	1.5382	3.8465
0.0033	0.1850	1.5382	3.8966
0.0029	0.1757	1.5382	3.9383
0.0027	0.1724	1.5382	3.9533
0.0025	0.1680	1.5382	3.9741
0.0022	0.1615	1.5382	4.0054
0.0020	0.1559	1.5382	4.0331
0.0018	0.1510	1.5382	4.0581

Consequently, defect may be found in terms of motor or cutter. The resultant situation may lead to model mixture distribution. Single distribution may lead to misinterpretation of the process monitoring inferences. To overcome this problem, we develop an alternative monitoring procedure named as MMCQ control chart based on the distribution function of a two component mixture of *Maxwell* distribution in which we wish to accommodate the proportion of each sub-populations according to its defect rate.

Let us assume that the occurrence of non-conformities produced from two sub population with expected value $\mu_1' = 2\sigma_i\sqrt{2/\pi}$; i=1, 2 and R represents the cumulative quantity of product inspected between two faulty items. Then R follows two components *Maxwell* mixture distribution. Now, recall the finite *Maxwell* mixture distribution given in equation (3.13) for k=2.

$$f_{MixMaxwell}(r)=\sum_{i=1}^{2}\sqrt{\frac{2}{\pi}}\frac{\theta_i}{\sigma_i^3}r^2e^{-\frac{r^2}{2\sigma_i^2}}, \tag{6.17}$$

and corresponding CDF

$$F_{MixMaxwell}(r)=\sum_{i=1}^{2}\theta_i\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{r^2}{2\sigma_i^2}\right). \tag{6.18}$$

Let us suppose that $p=\gamma\left(\frac{3}{2},\frac{r^2}{2\sigma_1^2}\right)$ and $q=\gamma\left(\frac{3}{2},\frac{r^2}{2\sigma_2^2}\right)$ such that $p=e^cq$. Now by substituting these (6.18) can be written as

$$\begin{aligned}F_{MixMaxwell}(r)&=\theta_1\frac{2}{\sqrt{\pi}}p+\theta_2\frac{2}{\sqrt{\pi}}q\\&=\frac{2}{\sqrt{\pi}}q\left(\theta_1e^c+\theta_2\right).\end{aligned}$$

$$\text{So, } F_{MixMaxwell}(r)=\frac{2}{\sqrt{\pi}}q\left[\theta_1\left(e^c-1\right)+1\right]. \tag{6.19}$$

Here, $\theta_1+\theta_2=1$. Hence, the α th quantile can be written as

$$R_\alpha=\sigma_2\sqrt{2F_{gamma}^{-1}\left\{\frac{\alpha}{\theta_1\left(e^c-1\right)+1}\right\}}. \tag{6.20}$$

To obtain two sided MMCQ control chart for fixed false alarm rate α we have to equate equation (6.19) equal to $\alpha/2$ for *LCL*, equal to $1-(\alpha/2)\,UCL$ and equal to $1/2$ for *CL*. Hence the final expression for these limits can be shown as underneath.

$$LCL: \quad R_L=\sigma_2\sqrt{2F_{gamma}^{-1}\left\{\frac{\alpha}{2}\frac{1}{\left(\theta_1e^c+\theta_2\right)}\right\}},$$

$$CL: \quad R_C=\sigma_2\sqrt{2F_{gamma}^{-1}\left\{\frac{1}{2}\frac{1}{\left(\theta_1e^c+\theta_2\right)}\right\}}, \text{and}$$

$$UCL: \quad R_U=\sigma_2\sqrt{2F_{gamma}^{-1}\left\{\left(1-\frac{\alpha}{2}\right)\frac{1}{\left(\theta_1e^c+\theta_2\right)}\right\}}.$$

These limits can also be presented as

$$LCL: \quad R_L = \sigma_2 \psi_L, \; CL: \quad R_C = \sigma_2 \psi_C \text{ and } UCL: \quad R_U = \sigma_2 \psi_U, \tag{6.21}$$

where,

$$\psi_L = \sqrt{2F^{-1}_{gamma}\left\{\frac{\alpha}{2}\frac{1}{\left(\theta_1 e^c + \theta_2\right)}\right\}},$$

$$\psi_C = \sqrt{2F^{-1}_{gamma}\left\{\frac{1}{2}\frac{1}{\left(\theta_1 e^c + \theta_2\right)}\right\}}, \text{ and}$$

$$\psi_U = \sqrt{2F^{-1}_{gamma}\left\{\left(1-\frac{\alpha}{2}\right)\frac{1}{\left(\theta_1 e^c + \theta_2\right)}\right\}}.$$

The one sided limits may also be presented in a similar fashion. To avoid complexity of calculation the ψ coefficient's values are given in the following table for different values of α, θ and c. The **Table 6.7**, shows the ψ coefficient's values for fixed false alarm rate α =0.005. The mixing proportion θ is given here for illustration purpose. other combination may also be possible for mixing proportion θ. Similarly, **Table 6.8**, displays the ψ coefficient's values for fixed false alarm rate α =0.0027 and **Table 6.9** depicts the ψ coefficient's values for fixed false alarm rate α =0.002.

Table 6.7 ψ coefficient for α =0.005

c	θ_1	θ_2		θ_1	θ_2		θ_1	θ_2	
	0.5	*0.5*		*0.6*	*0.4*		*0.7*	*0.3*	
	ψ_L	ψ_U	ψ_C	ψ_L	ψ_U	ψ_C	ψ_L	ψ_U	ψ_C
0.00	0.212	3.784	1.538	0.212	3.784	1.538	0.212	3.784	1.538
0.20	0.205	2.492	1.453	0.203	2.417	1.438	0.202	2.353	1.423
0.40	0.197	2.156	1.369	0.194	2.076	1.343	0.192	2.008	1.319
0.60	0.189	1.929	1.289	0.185	1.848	1.255	0.182	1.779	1.224
0.70	0.185	1.836	1.250	0.181	1.754	1.213	0.177	1.686	1.180
0.80	0.181	1.751	1.211	0.176	1.670	1.172	0.172	1.603	1.137
1.00	0.172	1.602	1.137	0.167	1.523	1.094	0.163	1.457	1.057
1.50	0.151	1.305	0.966	0.145	1.234	0.920	0.140	1.176	0.882

Table 6.8 ψ coefficient for α =0.0027

c	θ_1	θ_2		θ_1	θ_2		θ_1	θ_2	
	0.5	0.5		0.6	0.4		0.7	0.3	
	ψ_L	ψ_U	ψ_C	ψ_L	ψ_U	ψ_C	ψ_L	ψ_U	ψ_C
0.00	0.172	3.953	1.538	0.172	3.953	1.538	0.172	3.953	1.538
0.20	0.166	2.496	1.453	0.165	2.421	1.438	0.164	2.357	1.423
0.40	0.160	2.159	1.369	0.158	2.079	1.343	0.156	2.010	1.319
0.50	0.157	2.037	1.329	0.154	1.955	1.298	0.152	1.886	1.271
0.60	0.154	1.931	1.289	0.151	1.849	1.255	0.148	1.780	1.224
0.70	0.150	1.837	1.250	0.147	1.756	1.213	0.144	1.687	1.180
0.80	0.147	1.752	1.211	0.143	1.671	1.172	0.140	1.604	1.137
0.90	0.144	1.675	1.173	0.140	1.595	1.132	0.136	1.528	1.096
1.00	0.140	1.603	1.137	0.136	1.524	1.094	0.132	1.458	1.057
1.50	0.123	1.306	0.966	0.118	1.235	0.920	0.114	1.176	0.882

Table 6.9 ψ coefficient for α =0.002

c	θ_1	θ_2		θ_1	θ_2		θ_1	θ_2	
	0.5	0.5		0.6	0.4		0.7	0.3	
	ψ_L	ψ_U	ψ_C	ψ_L	ψ_U	ψ_C	ψ_L	ψ_U	ψ_C
0.00	0.156	4.033	1.538	0.156	4.033	1.538	0.156	4.033	1.538
0.20	0.150	2.498	1.453	0.149	2.423	1.438	0.149	2.358	1.423
0.40	0.145	2.159	1.369	0.143	2.079	1.343	0.141	2.011	1.319
0.50	0.142	2.037	1.329	0.140	1.956	1.298	0.138	1.887	1.271
0.60	0.139	1.932	1.289	0.136	1.850	1.255	0.134	1.781	1.224
0.70	0.136	1.838	1.250	0.133	1.756	1.213	0.130	1.688	1.180
0.80	0.133	1.753	1.211	0.130	1.672	1.172	0.127	1.604	1.137
0.90	0.130	1.675	1.173	0.126	1.595	1.132	0.123	1.528	1.096
1.00	0.127	1.603	1.137	0.123	1.524	1.094	0.120	1.458	1.057
1.50	0.111	1.306	0.966	0.107	1.235	0.920	0.103	1.176	0.882

The cumulative quantity between events (non-conformities or defects) R is plotted against the sample number in the MMCQ control charting system. If any non-conformity has been found R is restarted from zero. As long as the plotting statistic R stays within the control limits, the process is in control state. Whenever it crosses UCL that means there is some improvement in the process. On the other hand, if a point plotted below the LCL this is a signal that the process is deteriorating. In both these cases some preventive measure and corrective action are essential to make the process stable.

Corollary 6.2 In MMCQ control chart if $c = 0$ then it reduces to MCQ control chart. Hence, MMCQ is a generalization of MCQ. Alternatively, we can say that MCQ control chart is a special case of MMCQ control chart.

Corollary 6.3 The asymptotic behavior is given as: when $\sigma_2 \to 0$ then $R_L = R_C = R_U \to 0$ and when $\sigma_2 \to \infty$ then $R_L = R_C = R_U \to \infty$.

6.2.2. Performance evaluation using *ARL*

To check the performance of a chart different statistical tools are available in statistical process monitoring. In this section we discuss the most prominent statistical tool that measures the performance of a control chart named *ARL*. The *ARL* of the chart is defined as the average number of samples before we receive an out of control signal. It is expected that in a control situation *ARL* (i.e. ARL_0) is larger whereas *ARL* is lower in out of control situations (i.e. ARL_1).

Theorem 6.3 The one sided lower and upper structure of *ARLs* and two sided structure of *ARL* for MMCQ control chart are respectively as

$$ARL_L = \left[\alpha_L^{\delta}\left\{\theta_1\left(e^{c}-1\right)+1\right\}^{(1-\delta)}\right]^{-1},$$

$$ARL_U = \left[1-\left(1-\alpha_U\right)^{\delta}\left\{\theta_1\left(e^{c}-1\right)+1\right\}^{(1-\delta)}\right]^{-1}, \text{ and}$$

$$ARL_{L\&U} = \left[1+\left\{\alpha_L^{\delta}-\left(1-\alpha_U\right)^{\delta}\right\}\left\{\theta_1\left(e^{c}-1\right)+1\right\}^{(1-\delta)}\right]^{-1}.$$

Here, $\delta = \dfrac{\sigma_2^2}{\sigma_{02}^2}$ is the amount of shift in the process to be detected and α_L .and α_U are probability of false alarm for *LCL* and *UCL* respectively.

Proof. By the definition of *ARL,* in terms of *LCL* the for fixed alarm rate α_L

$$ARL_L = \frac{1}{\alpha_L} = \frac{1}{F_{MixMaxwell}\left(R_L\right)}.$$

Now, getting values from (6.19) we have,

$$ARL_L = \frac{1}{\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{R_L^2}{2\sigma_2^2}\right)\left\{\theta_1\left(e^c-1\right)+1\right\}}.$$

Now, by putting $R_L = \sigma_2\sqrt{2F_{gamma}^{-1}\left\{\frac{\alpha_L}{\theta_1\left(e^c-1\right)+1}\right\}}$ in the above equation we will get ARL_L

. Similarly, by substituting $R_U = \sigma_2\sqrt{2F_{gamma}^{-1}\left\{\frac{1-\alpha_U}{\theta_1\left(e^c-1\right)+1}\right\}}$ in

$$ARL_U = \frac{1}{1-F_{MixMaxwell}\left(R_U\right)},$$

or, $ARL_U = \dfrac{1}{1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{R_U^2}{2\sigma_2^2}\right)\left\{\theta_1\left(e^c-1\right)+1\right\}}$,

and by substituting both R_L and R_U in

$$ARL_{L\&U} = \frac{1}{F_{MixMaxwell}\left(R_L\right)+1-F_{MixMaxwell}\left(R_U\right)},$$

or, $ARL_{L\&U} = \dfrac{1}{1+\frac{2}{\sqrt{\pi}}\left\{\theta_1\left(e^c-1\right)+1\right\}\left\{\gamma\left(\frac{3}{2},\frac{R_L^2}{2\sigma_2^2}\right)-\gamma\left(\frac{3}{2},\frac{R_U^2}{2\sigma_2^2}\right)\right\}}$,

we will get the expression of *ARLs* given in the theorem. □

To compare the performance of MMCQ control chart with the MCQ control chart we plotted ARLs for different choices of false alarm rate, mixing proportion and *c*. For better looking we plotted $lnARL$ against e^{δ} in all **Figure 6.8**, **Figure 6.9** and **Figure 6.10**. The **Figure 6.8** (a)-(d) shows the ARL_L for α_L =0.0005, 0.00135, 0.005 and 0.05 and for θ_1 =0.4. Same amount of α's and θ's have been also taken for both AR_U and $AR_{L\&U}$ in the **Figure 6.9**(a)-(d) and **Figure 6.10** (a)-(d). We observe that when *c* = 0 then MMCQ

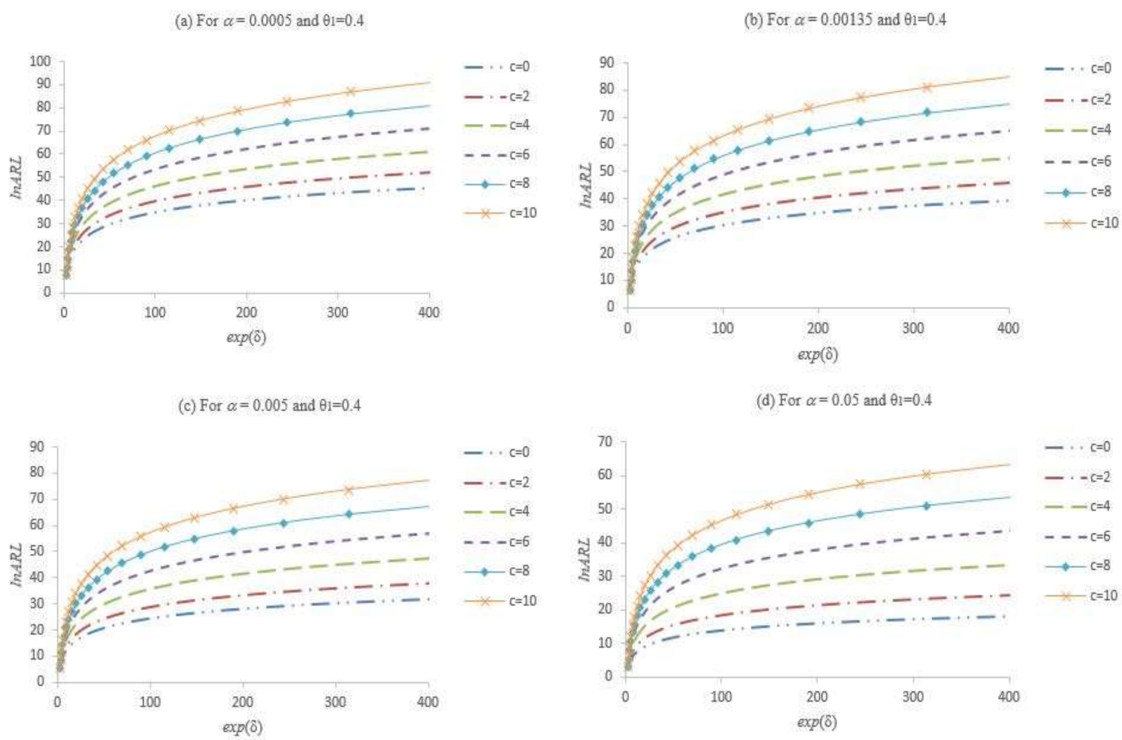


Figure 6.8: One sided lower ARL plots.

control chart behaves similar as usual MCQ control chart. Also it is obvious from figures that with the increase of the value of α the detection ability of out of control signal increases. In the **Figure 6.8** we see that when we increase the value of *c* the detection ability out of control signal is delayed. But in case of **Figure 6.9** and **Figure 6.10**, the opposite behavior is observed.

Another most important property of ARL is unbiasedness. We observe from the figures that when the process is deteriorating that is in the case of lower ARL_L the ARL is biased whereas the ARL is unbiased when the process is improving and both deteriorating and improving that is in case the of $AR\ _U$ and $ARL_{L\&U}$.

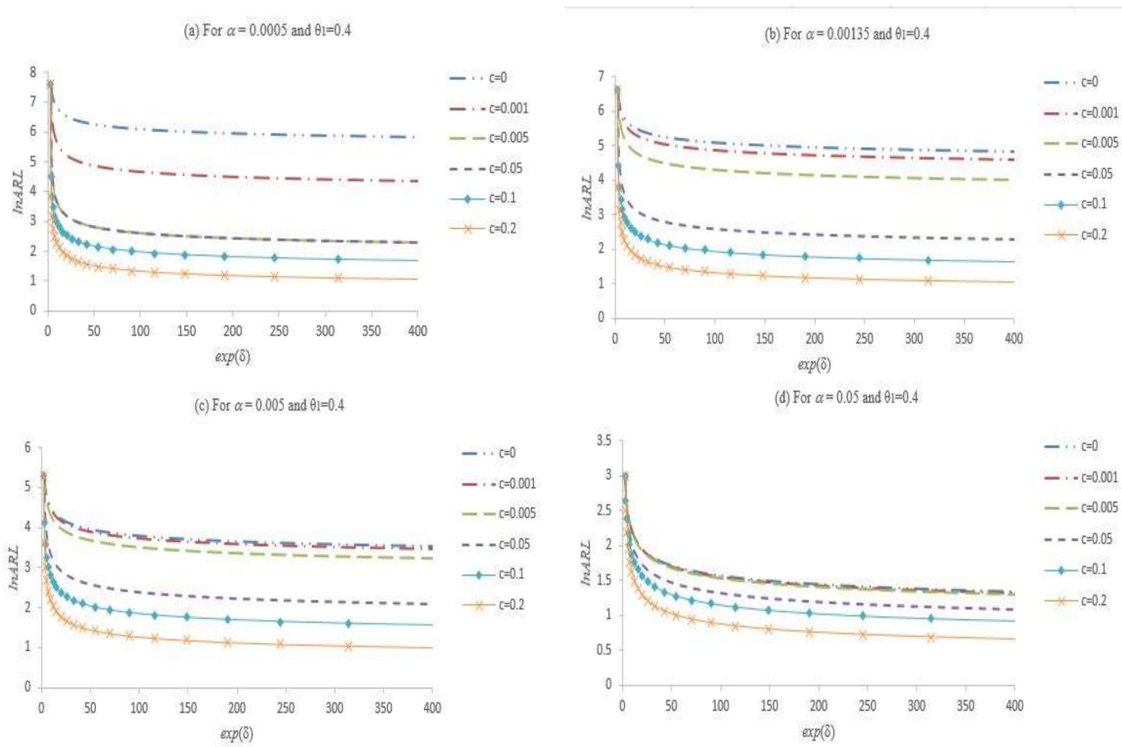


Figure 6.9: One sided upper *ARL* plots.

Finally, we can say that the detection ability of out of control signal of MMCQ control chart is better as compared to usual MCQ control chart when the process is improving and both improving and deteriorating.

6.2.3. An example to illustrate the chart in real life

In this section we provide an example to illustrate the proposed MMCQ control chart in the real life scenario. For the purpose of illustration we use the terminologies discussed in [69]. Let us, suppose that samples of size 300 meters are witnessed continuously and the witness process continues until any defect is found. Let us, for known $\sigma_{02} = 1952.20$ we constructed the control limits. For $\alpha = 0.0027$, $c = 0.2$ and $\theta_1 = \theta_2 = 0.5$ the ψ coefficient's can be obtain from the **Table 6.8** as $\psi_L = 0.1665$, $\psi_U = 2.4963$ and $\psi_C = 1.4527$. Hence, two sided control limits for MMCQ control chart are *LCL*=324.07, *CL*=2836.55 and *UCL*=

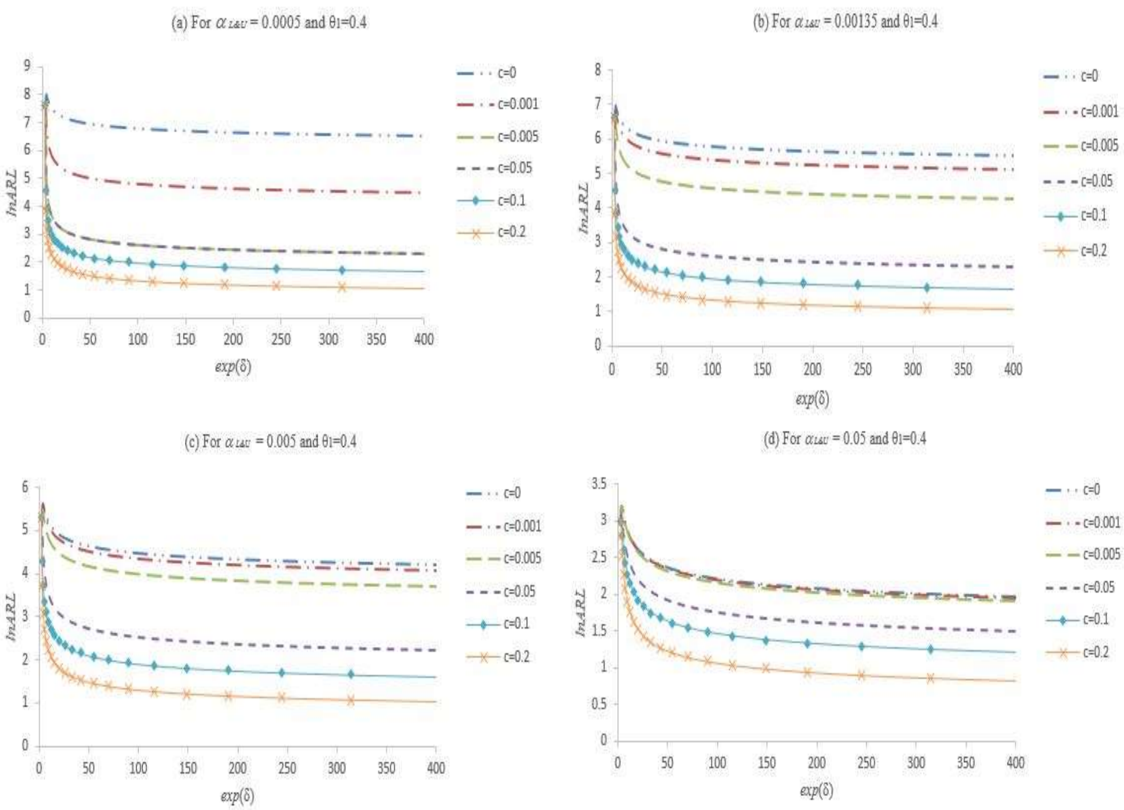


Figure 6.10: Two sided *ARL* plots.

4872.69. Again, for $\alpha = 0.0027$ the ω coefficients can be obtained from **Table 6.6** as $\omega_L = 0.1724$, $\omega_C = 1.5382$ and $\omega_U = 3.9533$. Consequently, the two sided control limits for MCQ control chart are *LCL*= 376.89, *CL*=2734.65 and *UCL*=6727.82.

Now, using the data simulated in section 6.1 we compared the MCQ and MMCQ control charts. The plot given in **Figure 6.11** is a situation when the process is in control. For both the MCQ and MMCQ control charts the plot shows that the process is in control. To examine the detection ability of signals MCQ and MMCQ control charts let us introduce a shift $\delta = 2.25$ after the 19th sample. The scenario is presented in the **Figure 6.12**. It is clear from the figure that some of the points after 19th sample went beyond the UCL of MMCQ control chart whereas all the points still in between LCL and UCL in MCQ control chart.

From these control limits we can say that the MMCQ performs better as compared to MCQ control chart.

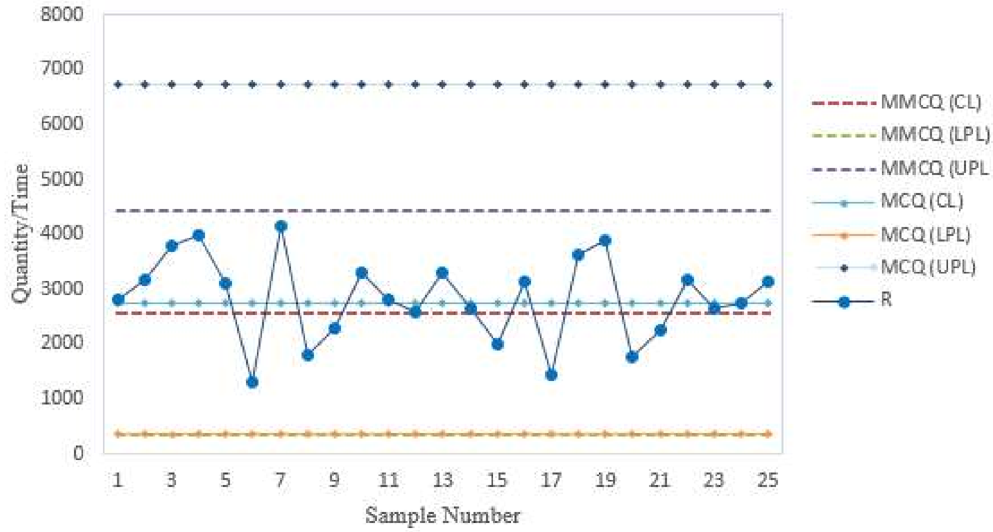


Figure 6.11: Comparison between MCQ and MMCQ control charts (in control)

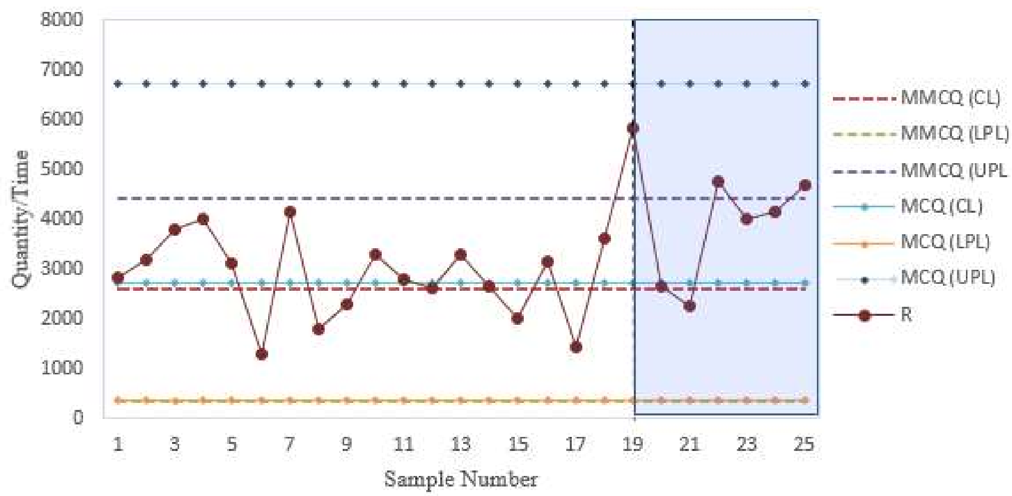
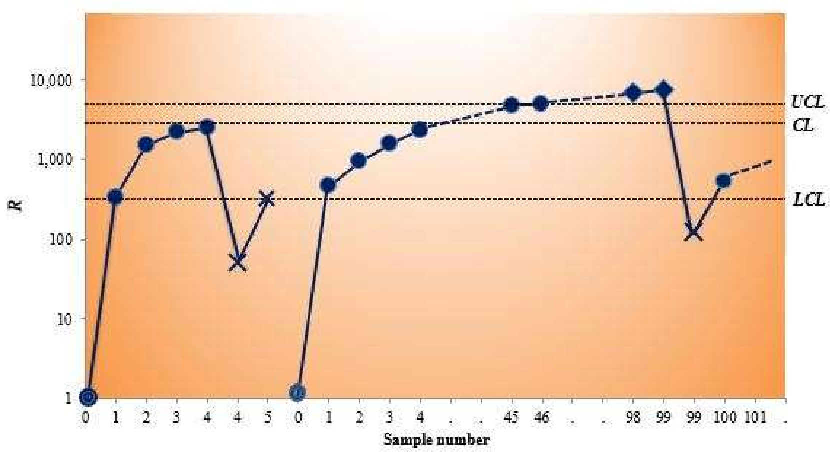


Figure 6.12: Comparison between MCQ and MMCQ control charts (out control control)

A hypothetical dataset is given in the following table for the above settings.

Sample Number	Defect Observed?	Cumulative Quantity inspected *R*	Indication	Reset to zero?
1	No	$330 \geq$ LCL	n.d.	No
2	No	$1500 \geq$ LCL	i.c.	No
3	No	$2200 \geq$ LCL	i.c	No

Sample Number	Defect Observed?	Cumulative Quantity inspected R	Indication	Reset to zero?
4	Yes	$2500 \geq$ LCL	i.c.	Yes
4	No	*50	n.d.	No
5	Yes	$320 \leq$ LCL	o.c.	Yes
1	No	$458 \geq$ LCL	n.d.	No
2	No	$950 \geq$ LCL	i.c.	No
3	No	$1580 \geq$ LCL	i.c.	No
4	No	$2340 \geq$ LCL	i.c.	No
⋮	⋮	⋮	⋮	⋮
45	No	$4850 \leq$ UCL	i.c.	No
46	No	$4920 \geq$ UCL	im.	No
⋮	⋮	⋮	⋮	⋮
98	No	$6890 \geq$ UCL	im.	No
99	Yes	$7500 \geq$ UCL	im.	Yes
99	No	*120	n.d.	No
100	No	$540 \geq$ LCL	n.d.	No
⋮	⋮	⋮	⋮	⋮



6.3. Application of finite mixture of *Maxwell* distribution in reliability engineering

Consider the data given earlier subsection 6.1.4 on vertical boring machine. Let us say a VBM is produced by two different machines in a company. Also, let the proportion produced from each machine is 60% and 40% respectively. Now if we randomly select a

VBM from the mixture of total produced machines in the company, then the probability that the strength of the VBM is less than r can be expressed as

$$P\left[R<r\right]=P\left[R<r\mid Machine1\right]P\left(Machine1\right)+P\left[R<r\mid Machine2\right]P\left(Machine2\right). \tag{6.22}$$

More compactly,

$$F\left(r\right)=0.60F_1\left(r\right)+0.40F_2\left(r\right). \tag{6.23}$$

And the mean strength of the VBM can be computed by the equation given as below using the *low of total expectation*.

$$E\left(R\right)=0.60E\left(R\mid Machine1\right)+0.40E\left(R\mid Machine2\right). \tag{6.24}$$

Equations (6.23) and (6.24) can also be generalized to a greater number of mixture components. Now, recall the VBM observation in following fashion

Machine 1	2802	2937	2136	4359
	4020	1781	2816	2655
	3886	2296	3158	3695
	4155	3811	2380	376
Machine 2	2172	3705	2848	4339
	2076	2672	3632	1976
	1700	1596	1701	3575
	3802	4351	4291	808

Let us suppose that the two distributions are *Maxwell* distribution with different scale parameters given as table below. Also the expected value and F (3000) for each component distribution have been given in the following table.

Distribution	Estimated Scale parameter (σ)	Proportion	$F\left(3000\right)$	$E\left(X\mid Machine\right)$
Machine 1	1074.77	0.60	0.94947	1715.092
Machine 2	1045.24	0.40	0.95866	1667.962

Hence, the exact value of $F\left(3000\right)$ and the expected value for the two components mixture of *Maxwell* distribution can be computed by using equations (6.23) and (6.24).

$F\left(3000\right)=0.60\times0.94947+0.40\times0.95866=0.95315.$

$E\left(X\right)=0.60\times1715.09+0.40\times1667.96=1030.26.$

Reliability is defined as the probability that a device will perform its intended function during a specified period of time under stated conditions. Hence, a VBM will persist 3000 hours or more, either it is from machine 1 or 2, is

$S\left(3000\right)=1-F\left(3000\right)=1-0.95315$=0.03032.

6.4. Application summary

Application is always recommended with any proposed technique. In this point of view, we provided different possible applications of mixture distributions. *Maxwell* distribution is a probabilistic distribution use widely in statistical mechanics, lifetime modelling, and chemistry [2]' [3]' [4]. But in the field of statistical process control it has not been studied yet. Again, in the field of process monitoring one of the stringent assumption is normal distribution that may not always be fulfilled in real processes. Hence as a non-normal skewed distribution we proposed *Maxwell* distribution in the field of statistical process control. We discussed some properties of *Maxwell* distribution. We then demonstrated how to construct control chart for the *Maxwell* parameter using pivotal quantity and provided some simulated results. In addition, we also provided a real life example that follows *Maxwell* distribution where this control chart might be used to judge the stability of the process to be monitored.

In addition to monitoring *Maxwell* process parameter using typical Shewhart method, we also proposed an alternative method to monitoring a process of non-conformities using mixture *Maxwell* distribution. This control charting technique is named as the MMCQ control chart. In this technique, time between defects is the cumulative quantity which we considered to follows transformed mixture *Maxwell* distribution. If any defective items are detected, then the cumulative plotting statistic needs to be restarted from zero. Some practical scenario in VBM manufacturing industry have been presented to illustrate the MMCQ control charting method.

Moreover, the application of mixture *Maxwell* distribution in reliability engineering has also been discussed in this chapter. In the next chapter we will draw a final conclusion and proposed some recommendations regarding finite and infinite mixture distribution, particularly on the *Maxwell* mixture distributions.

CHAPTER SEVEN

CONCLUSION AND RECOMMENDATIONS

Maxwell distribution has been studied in different flavor in the current thesis. We developed two different types of mixture distribution from *Maxwell* distribution namely finite and infinite mixture of *Maxwell* distributions. k component mixture of *Maxwell* distribution is presented as an example of finite mixture and *tau square* mixture of *Maxwell* distribution is presented as an example of infinite mixture of *Maxwell* distribution. Different properties of these two distribution are also studied. Estimation methods has been discussed separately. Finally, simulation study and real life example have been discussed in this thesis.

Mixture distributions particularly *Maxwell* mixture distributions has been studied/modeled in the literature for two components or subpopulations under type I censoring. But the behavior of this finite mixture distribution under complete data has not yet been studied. Hence in this thesis, we studied the properties of the model for complete sample. In addition, the literature has investigations of a mixture of only two *Maxwell* subpopulations. In the current thesis we extended the work to k subpopulations. As an application, in the field of process monitoring we studied the *Maxwell* distribution. Beside this *Maxwell* mixture distribution is implemented in cumulative quantity control charting system. In engineering application, we investigated the reliability of *vertical boring machine* (VBM) after a certain amount of time. However, the problem is complicated by the fact that the parent population has a distribution that is composed of distinct subpopulations that can best be modeled by a k finite mixture *Maxwell* distribution.

In Chapter three, we developed different properties of finite mixture distribution in general. We implemented these properties for k component mixture of *Maxwell* distribution. The MGF and characteristic function (CF) of mixture of *Maxwell* distribution is the weighted MGF and CF of the component distributions. First four moments of the finite mixture of *Maxwell* distribution have been studied as well. We considered some special cases of the proposed distribution which may be implemented in some real life scenarios.

In Chapter four, we introduced a new distribution called *Tau square* distribution which has similar properties as *chi square*. When $\mathrm{T}^2 = X/\nu$ and $\nu = 3n$ then *tau square* distribution reduces to chi square distribution with ν degrees of freedom. The distribution is derived from the MLE of *Maxwell* distribution. This distribution has been mixed with the *Maxwell* distribution. In the Chapter, we mainly focused on different characteristics of *tau square* mixture of *Maxwell* distribution. Among these properties MGF, CF, survival function and hazard function are specifically discussed. Beside this, the m-th raw moments of the distribution is also discussed.

As analytical solution of MLE estimation is difficult for finite *Maxwell* distribution, we estimated the parameters of the distribution using the EM algorithm in Chapter five. Another well-known method of estimation is method of moment, which we implemented for the *tau square* mixture of *Maxwell* distribution.

In Chapter six (the application chapter), we applied *Maxwell* distribution in statistical process control. In this field of research many distributions have been used before, but *Maxwell* distribution has not been applied yet. In addition to monitoring *Maxwell* process parameter using typical Shewhart method, we also proposed Mixture *Maxwell* Cumulative Quantity (MMCQ) control chart to monitor a process of non-conformities using mixture *Maxwell* distribution. Some practical scenario in VBM manufacturing industry have been presented to illustrate the MMCQ control charting method. Moreover, the application of mixture *Maxwell* distribution in reliability engineering has also been discussed in the Chapter six.

7.1. Recommendations

In this section we would like to recommend the use of Maxwell mixture distribution for further future research. Some recommendations are given underneath.

- We studied finite Maxwell mixture distribution with some specific properties. We haven't examined some other important properties such as limiting distribution etc.
- In this thesis, we have looked at only MLE and method of moments. We may also consider in the future other methods of estimation such as the Bayesian estimation.
- Infinite *Maxwell* distribution is discussed for only *tau square* distribution. What if we use other distribution instead of *tau square* distribution?
- In the application part, we discussed control chart named Shewhart control chart. The EWMA and CUSUM structures of control chart can also be implemented for the Maxwell distribution.
- Can we apply these mixture distributions in other field of research?

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